Numerical Integration

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Numerical integration is a technique which can be used to solve integrals when a closed, algebraic solution cannot be found. The method has the same initial steps as setting up an integral, and can help make that process more understandable by making more concrete the steps involved. The specific problem this will be applied to is the electric field of distributed charges.

All these problems have the same strategy:
• Identify the observation (field) point, the location where the electric field is to be found.
• Split the object into some number of (small) pieces.
• Assume that the pieces can be treated as point charges.
• Find the the electric field for each piece at that point using Coulomb’s Law: \[ \Delta \vec{E} = \frac{k \Delta Q}{r^2} \hat{r} \]. To do this, the charge of each piece and the vector from each piece to the observation point must be found.
• Find the components of the electric field in the coordinate directions.
• For each component, add all the contributions from all the pieces together to get the net field.

The equation for the electric field is the same as that of a point charge. The only difference is the \( \Delta \) in front of the \( Q \) and the \( E \). One meaning of \( \Delta \) is the change in a quantity; however the meaning used here is that of the contribution of a (small) piece to the whole. \( \Delta E \) is the magnitude of the field created by a small piece. When all the fields from all the pieces are added together, the net field is found. \( \Delta Q \) is the amount of charge on a piece. Just as with the field, when all the \( \Delta Q \) are added together, the total charge \( Q \) is the result. It is this sense in which \( \Delta \) is used, and this is the link later to integrals.

The first five steps are the same whether setting up an integral or doing a numerical integration. The last step is the part that requires the aid of a computer if the number of pieces is large. The difficult part is determining expressions for \( \Delta Q \), and for \( \hat{r} \), the vector connecting the source piece to the field point. As an example, the electric field for a rod of charge will be found anywhere around it. The special case of the observation point being along the perpendicular bisector is done exactly in the text (on p. 690), but the general problem cannot be solved so easily.
The electric field due to a uniform rod

The configuration of the rod is the same as that in the book. The rod has a total length $L$ and a total charge $Q$ distributed uniformly along it. The rod is assumed to be thin so that the width of it can be neglected. It is oriented along the $y$-axis with the center at the origin. The $x$-axis is chosen to point in the direction parallel from the rod to the point of observation $P$. This simplifies the problem allowing the $z$-axis and components to be ignored (due to the symmetry of the problem). The coordinates of the observation point are $(X_p, Y_p)$.

The rod can be split into pieces and each piece can be treated as a point charge. If the pieces are small enough, this is a good approximation and the results will be very close to the true value. However, the technique can be done even if there are not many pieces and the pieces are not small; the resulting solution will not be as good an approximation, though. For illustration, the rod will be split up into 4 pieces. However, in the derivations to keep it general, the number of pieces will be $N$.

Find the amount of charge on each piece

Each of the pieces is assumed to be a point charge at its center. The amount of charge on each one, $\Delta Q$, needs to be known. Each piece has a length which is $1/N$ times the total length. Call this length $\Delta y= L/N$. Since the charge is assumed to be uniformly distributed along the rod, each piece has a charge on it $(\Delta Q)$ which is the same fraction of the total charge as the length of that piece is to the total length. That is $\Delta Q / Q = \Delta y / L$, or $\Delta Q = Q \Delta y / L$. This gives the $\Delta Q$ needed for Coulomb’s Law.

Find the distance between each piece and the observation point

In order to find the distance to the observation point, the location of each piece needs to be known. If they are numbered 1 to $N$ starting from the bottom end, the $y$ coordinate of the position is $y_i$, where $i$ is the piece number. For all of them, $x_i = 0$. The $y$ coordinate of the bottom edge of any piece is given by $-L/2 + (i-1)\Delta y$. $-L/2$ is the bottom edge of the bottom-most piece, and each successive one is $\Delta y$ above the previous. The center is another $\Delta y / 2$ above that, so $y_i = -L / 2 + (i - 1)\Delta y + \Delta y / 2$. Since $\Delta y = L/N$, we can write $y_i = -L / 2 + (i - 1)L / N + L / 2 N$. So we now have an expression for $y_i$, the $y$ position of the $i$th piece, in terms only of the number of the piece $i$, the length of the rod $L$, and the number of pieces we are splitting the rod into $N$. The vector between the center of the piece and the observation points is $\hat{r}_i = (x_p - x_i)\hat{i} + (y_p - y_i)\hat{j}$, or just $\hat{r}_i = (x_p)\hat{i} + (y_p - y_i)\hat{j}$, since $x_i = 0$ for each piece; note that $\hat{r}_i$ is just $\frac{\hat{r}_i}{|\hat{r}_i|}$.
The electric field due to each piece can now be written since the charge and vector connecting the charge to the field point are known: \[ \Delta \vec{E}_i = k \frac{Q}{r_i^2} \hat{r}_i = k \frac{Q \Delta y / L}{x_p^2 + (y_p - y_i)^2} \frac{x_p}{\sqrt{x_p^2 + (y_p - y_i)^2}} (x_p \hat{i} + (y_p - y_i) \hat{j}). \]

Find the components of the electric field

With the contribution to the electric field from some piece of the charge written as above, it is trivial to get the components of the electric field in each direction:

\[ \Delta E_{i,x} = k \frac{Q \Delta y / L}{x_p^2 + (y_p - y_i)^2} \frac{x_p}{\sqrt{x_p^2 + (y_p - y_i)^2}} \frac{kQ}{N} \left( \frac{x_p}{x_p^2 + (y_p - y_i)^2} \right)^{3/2} \]
\[ \Delta E_{i,y} = k \frac{Q \Delta y / L}{x_p^2 + (y_p - y_i)^2} \frac{y_p - y_i}{\sqrt{x_p^2 + (y_p - y_i)^2}} \frac{kQ}{N} \left( \frac{y_p - y_i}{x_p^2 + (y_p - y_i)^2} \right)^{3/2}. \]

Note that the form above guarantees that the signs of all the components will be correct: for example if \( y_p < y_i \) (as it is for piece #4 shown above), and if the charge is positive, the y component of the electric field should be negative, which it is, since \( y_p < y_i \rightarrow y_p - y_i < 0 \). If the charge were negative, then the y component of the field for this case should be positive, which the negative sign on the charge would account for.

We developed previously that \( y_i = -L / 2 + (i - 1)L / N + L / 2N \). Since \( k \) is a constant, \( Q \) and \( L \) are fixed parameters of the rod, and the coordinates of the observation point \((x_p, y_p)\) are also fixed, our expressions for the components of the electric field from the ith piece of the rod is only a function of i.
Adding the contributions

The net field is found by summing the components of each piece together. Putting our previous equations together gives:

\[ E_x = \sum_{i=1}^{N} \Delta E_{i,x} \]
\[ = \frac{kQ}{N} \sum_{i=1}^{N} \frac{x_p}{(x_p^2 + (y_p - y_i)^2)^{3/2}} \]

and

\[ E_y = \sum_{i=1}^{N} \Delta E_{i,y} \]
\[ = \frac{kQ}{N} \sum_{i=1}^{N} \frac{y_p - y_i}{(x_p^2 + (y_p - y_i)^2)^{3/2}} \]

where \( y_i = -L / 2 + (i - 1)L / N + L / 2N \).

Exercise #1: Consider a rod of length 1 m with charge 10 \( \mu \)C uniformly distributed along its length. Split the rod into four equal pieces and determine the electric field at the point (0.4 m, 0.0 m) using the relationships determined above. Do the calculations by hand (or hand calculator, but not computer) to get a sense for what is going on.

Exercise #2: The situation in Exercise #1 is finding the electric field along the perpendicular bisector of a finite line segment. We have developed an exact solution to this problem (eq. 23-8 in the text). Use the exact solution to calculate the electric field for the situation in Exercise #1, and compare to your result for Exercise #1.

Exercise #3: Develop a spreadsheet (or computer program) that will automate the calculations above. Write a program or spreadsheet that will calculate the electric field anywhere in the x-y plane for a finite rod of fixed uniform charge (for locations not on the rod). (To be completed as Computer Exercise #1).

Exercise #4: (Optional, if time). Write a spreadsheet (or computer program) that will calculate the electric field due to a two-dimensional finite square plate of charge that lies in the x-y plane with one corner at the origin. The program should be able to calculate the electric field anywhere in three-dimensional space (except perhaps for points within the charge distribution itself). Note that this can also be done analytically with little difficulty; it just becomes a double integral as you need two coordinates to specify the location of the charge.

Exercise #5: (Bonus: can replace your lowest homework score). Write a spreadsheet (or computer program) that will calculate the electric field due to a uniform ring of charge. The program should be able to calculate the electric field anywhere in three-dimensional space (except perhaps for points lying on the charge distribution itself). Note that the text does the special case of the field on the axis of the ring; the general case of off the axis is much harder to do analytically. Even though this is a ring, you still only need one counting variable (hint: think about polar coordinates).