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Simple soluble models of quantum damping applied to cavity quantum electrodynamics

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We present two simple soluble quantum optical models in which an atomic system is coupled to a bath containing an infinite number of reservoir states. The total system is treated conservatively, but the atomic subsystem of interest exhibits decay. The first system consists of a single two-level atom coupled to an infinite number of modes of a one-dimensional cavity which are equally spaced in frequency. The coupling results in spontaneous emission by an excited atom into the modes of the cavity. The model also exhibits the subsequent re-excitation of the atom after the spontaneously emitted radiation is reflected from the ends of the cavity. The second is a single two-level atom coupled to a single mode of a damped optical cavity where the damping is provided by coupling to quantized oscillators in the walls of the cavity. The damped cavity alters the spontaneous emission rate from its free space value. The simplicity of both models makes them attractive as a method for introducing quantum mechanical damping phenomena.

I. INTRODUCTION

Damping phenomena are ubiquitous in physics. Canonical quantization procedures are, however, strictly applicable only to conservative systems. The usual approach to quantum mechanical damping involves the coupling of a system of interest to a bath or reservoir consisting of a large number of particles or modes of a field. The entire system consisting of the subsystem of interest plus the bath is then treated conservatively. In the limit of a large number of degrees of freedom in the bath, the dynamical variables of the subsystem of interest may then exhibit irreversible decay, or damping. Such problems can often be simplified by removing uninteresting information about the reservoir states using density matrix techniques. This approach has been used in systems of varying levels of complexity.[1–3] In quantum optics, spontaneous emission is perhaps the most fundamental manifestation of damping. In this paper, we present two simple soluble models of damped atomic excitation in which the entire system can be treated using elementary quantum mechanics. Both models have been treated previously using other techniques, but the simplicity of our methods should be of interest to readers of this journal.

The first system we consider is the well-known Wigner–Weisskopf model of spontaneous emission[4] in which an excited two-level atom is coupled to the empty modes of the radiation field of a one-dimensional cavity, i.e., a cavity at temperature $T=0$. Using elementary techniques, we derive exact expressions for the time evolution for all variables of the system, including the decay of the atomic excitation. We also note and interpret periodic revivals of the atomic excitation. The price for the ease of derivation is that the formulas we arrive at are not as transparent as the usual expressions describing such damping, although they are easy to evaluate with the aid of a personal computer. Our result is an extension of an often neglected portion of Wigner and Weisskopf's original paper. This extension is now possible due to the advent of computers. The second model we consider is a single two-level atom coupled to a single mode of a damped optical cavity. Using the same techniques that we develop in solving the Wigner–Weisskopf model, we are able to relate

the Q of the cavity to the fundamental properties of the coupling between photons in the cavity and the quantized modes of excitations in the mirrors of the cavity, and then derive expressions which demonstrate the effect of the properties of the cavity on the decay of the atomic excitation.

II. DECAY AND REVIVAL OF AN ATOM IN A ONE-DIMENSIONAL CAVITY

Perhaps the simplest model for the decay of a quantum mechanical system is that first given by Wigner and Weisskopf[4] in which an excited state of a single two-level atom (with resonance frequency ω_0) is coupled to an infinite number of empty modes of the radiation field (with frequencies ω_i). In the simplest case the couplings of the atom to the radiation modes which comprise the reservoir are all equivalent. The Hamiltonian for this system[5,6] is given by

$$H=H_{\text{atom}}+H_{\text{field}}+H_{\text{interaction}}$$
$$=\hbar\omega_0 b^\dagger b+\sum_i \hbar\omega_i(a_i^\dagger a_i+1/2)+\sum_i \hbar\Omega_i(a_i^\dagger+a_i)$$
$$\times(b^\dagger+b), \tag{1}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operators respectively for photons in the mode specified by i, and $b^\dagger$ and b are the raising and lowering operators for the two level atomic excitation, i.e., if the atomic ground state and excited state are represented by $|g\rangle$ and $|e\rangle$ respectively, then

$$b=|g\rangle\langle e| \quad \text{and} \quad b^\dagger=|e\rangle\langle g|. \tag{2}$$

The magnitudes of the couplings of the photon modes to the atom are given by the constants Ω_i. In a one-dimensional cavity, the dipole approximation coupling parameter for the ith mode is given by[6]

$$\Omega_i=\left(\frac{\omega_0}{\hbar 2\epsilon_0 V}\right)^{1/2} f_i(r)d, \tag{3}$$

where $f_i(r)$ is a function with a maximum value of unity which describes the spatial dependence of the electric field in the mode, V is the effective volume of the mode given by $V=\int|f(r)|^2\,d^3r$, and d is the dipole matrix element between the two atomic levels, $|\langle e|q_e\mathbf{R}|g\rangle|$. The fact that the coupling constant does not depend on frequency is a result of the dipole approximation. There is, of course, a slight dependence on ω in better approximations. However, as Wigner and Weisskopf demonstrated,[4] and as we will see in this paper, the only photon modes that play a significant role in spontaneous emission are those that are very near the atomic resonance frequency. In the neighborhood of the resonance, the variation of the coupling constant with frequency is very slight.

We simplify the following calculations in two additional ways. First, we begin by assuming that the atom is located at the center of the cavity. At this position the field modes have either nodes or antinodes, which are described by values of f_i which are either 1, -1, or 0. For the atom at the center, we thus need only consider interactions of the atom with half of the cavity modes, and the coupling constant for all of the relevant modes is a constant, which we call Ω.

The second simplification we make is the standard rotating wave approximation[5] in which terms involving the products $b^\dagger a_i^\dagger$ and $a_i b$ are dropped from $H_{\text{interaction}}$. (We note that this approximation is exactly that made by Wigner and Weisskopf in their original 1930 paper,[4] although the expression "rotating wave approximation" was not used at that time.)

Although we will assume that the number of modes of the cavity becomes large, we emphasize that we do not assume that the spacing between the modes becomes small. In terms of our one-dimensional cavity this means that we do *not* assume that the length L necessarily goes to infinity, but we do assume that the atomic resonance frequency is large compared to the mode spacing, i.e., $\omega_0 \gg \pi c/L$. This insures that there are a large number of modes on either side of the resonance with which the atom will interact. Our model accommodates the continuum limit, if we let L go to infinity, as well as the case where the discreteness of the field energy levels is manifest in revivals of atomic excitations.

The combined states of the atom and field may be denoted by the atomic state (e or g) along with the number of photons in each mode of the field. As an example, the state in which the atom is in the excited state with no field photons can be written as $|e;\varnothing\rangle$, while a ground state atom with one photon in the "kth" mode of the field can be written as $|g;k\rangle$. These states are eigenstates of

$$H_0=H_{\text{atom}}+H_{\text{field}}, \tag{4}$$

but they are not eigenstates of the complete Hamiltonian including the interaction term. For convenience, we assume that the atom is interacting with a finite number of modes, and we will eventually take the limit as this number becomes very large. In a model one-dimensional cavity the modes will be equally spaced in angular frequency by $\Delta=2\pi(c/2L)=\pi c/L$. For an atom situated at the center of the cavity the effective spacing is twice this value because, as mentioned before, the atom only interacts with the modes which do not have nodes at this position. Therefore, $\Delta\rightarrow 2\pi c/L$. We assume that the central mode of our finite number is exactly resonant with the atomic frequency. Thus, the detunings of the various modes from the atomic resonance are given by $-n\Delta$, $-(n-1)\Delta,\ldots,0,\ldots n\Delta$, where we will initially consider $2n+1$ modes. The labelings of the modes in the kets $|g;j\rangle$ will correspond to the integer which describes the detuning of the mode in units of Δ.

In this rotating wave approximation model the state containing an excited atom and no photons, $|e;\varnothing\rangle$, is coupled to all of the states with a ground state atom and one photon, such as $|g;-n\rangle$, and these are the only nonzero couplings. We represent the state vectors with the following column vectors

$$|e;\varnothing\rangle\rightarrow\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix},\quad |g;-n\rangle\rightarrow\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix},$$
$$|g;-n+1\rangle\rightarrow\begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix},\quad \text{etc.} \tag{5}$$

If we subtract off the zero-point energy of the radiation modes plus the excited state atomic energy, the Hamiltonian of Eq. (1) can be represented by the matrix

$$\begin{pmatrix} 0 & \hbar\Omega & -\hbar\Omega & \hbar\Omega & \cdots & -\hbar\Omega \\ \hbar\Omega & -n\hbar\Delta & 0 & 0 & \cdots & \\ -\hbar\Omega & 0 & -(n-1)\hbar\Delta & 0 & & \\ \hbar\Omega & 0 & 0 & & & \\ \vdots & & & & \ddots & \\ \hbar\Omega & & & & & n\hbar\Delta \end{pmatrix}. \tag{6}$$

The time evolution governed by a Hamiltonian of this form has been calculated previously using at least two different approaches. A technique of Wigner and Weisskopf that has been summarized in many current texts[7,8] begins by writing the general state of the system as a linear combination of the states we have described above in Eq. (5) with time-dependent coefficients. Substitution of this general state vector into the Schrödinger equation results in an integrodifferential equation for the coefficient of the state $|e;\varnothing\rangle$. By letting the spacing between the modes of the field go to zero this equation can be solved using various approximations. A different approach using Laplace transform techniques has been developed by Stey and Gibberd[9] for the model Hamiltonian of Eq. (6), giving an exact solution. In the following paragraphs, we derive an exact solution in a different manner. The mathematics of our method is much simpler than that of Stey and Gibberd although the form in which our solution is expressed is not as transparent as theirs.

Our strategy is to find the exact representations for the eigenstates of our model Hamiltonian. Once the eigenstates are known, the initial state of the system is projected onto these eigenstates, and the time evolution of the system can be calculated easily because the time dependence of the eigenstates is trivial. In mathematical language, we find the eigenstates of the total Hamiltonian:

$$H|E_i\rangle = E_i|E_i\rangle. \tag{7}$$

The initial state of the system, with the atom excited and no photon in the cavity, can be written as a linear combination of these energy eigenstates

$$|\psi(0)\rangle = |e;\varnothing\rangle = \sum_i |E_i\rangle\langle E_i|e;\varnothing\rangle = \sum_i a_i|E_i\rangle. \tag{8}$$

At a later time t, the state of the system is given by

$$|\psi(t)\rangle = \sum_i a_i \exp(-iE_it/\hbar)|E_i\rangle, \tag{9}$$

and the amplitude for the atom to still be in the excited state is given by

$$\langle e;\varnothing|\psi(t)\rangle = \sum_i a_i \exp(-iE_it/\hbar)\langle e;\varnothing|E_i\rangle$$

$$= \sum_i |a_i|^2 \exp(-iE_it/\hbar). \tag{10}$$

The only difficulty in this procedure is the determination of the eigenvalues and the eigenstates of the complete Hamiltonian. The problem, in matrix representation, is to find the eigenvectors, $\tilde{x}^{(i)}$, and eigenvalues, E_i, of the matrix $\tilde{H}$ in Eq. (6):

$$\tilde{H}\tilde{x}^{(i)} = E_i\tilde{x}^{(i)}. \tag{11}$$

Writing out each of the $2n+2$ individual equations which are contained in Eq. (11) gives

$$\hbar\Omega(x_2^{(i)} - x_3^{(i)} + \cdots + x_{2n+2}^{(i)}) = E_i x_1^{(i)},$$

$$\hbar\Omega x_1^{(i)} - n\hbar\Delta x_2^{(i)} = E_i x_2^{(i)},$$

$$-\hbar\Omega x_1^{(i)} - (n-1)\hbar\Delta x_3^{(i)} = E_i x_3^{(i)},$$

$$\vdots$$

$$\hbar\Omega x_1^{(i)} + n\hbar\Delta x_{2n+2}^{(i)} = E_i x_{2n+2}^{(i)}, \tag{12}$$

where $x_j^{(i)}$ is the jth component of the eigenvector associated with energy E_i. Each of the lower $2n+1$ of these equations can be rewritten to find expressions for each of the $x_j^{(i)}$'s ($j>1$) in terms of $x_1^{(i)}$. These can than be inserted in the first equation. Upon cancellation of the common $x_1^{(i)}$ in all of the terms, we are left with the characteristic equation

$$\frac{1}{(E_i+n\hbar\Delta)} + \frac{1}{(E_i+(n-1)\hbar\Delta)} + \cdots + \frac{1}{(E_i-n\hbar\Delta)} = E_i/\hbar^2\Omega^2, \tag{13}$$

from which to determine the energies of the eigenstates.

For numerical work, it is easier to write in Eq. (13) terms of the dimensionless variable

$$y = E_i/\hbar\Delta, \tag{14}$$

so that the characteristic equation becomes

$$\frac{1}{(y+n)} + \frac{1}{(y+(n-1))} + \cdots + \frac{1}{(y-n)} = (\Delta/\Omega)^2 y. \tag{15}$$

In the limit of large n, the series on the left-hand side can be expressed simply,[10] and the characteristic equation becomes

$$\pi \cot(\pi y) = (\Delta/\Omega)^2 y. \tag{16}$$

This transcendental characteristic equation was developed by Weisskopf and Wigner,[4] but its utility to them was limited by the difficulty in finding and using the roots without a computer. Examination of the cotangent function makes it clear that there will be a root of this transcendental equation between each of the integers. It is a simple and quick matter using a personal computer to find such well-bracketed roots to arbitrary precision.

Once the energies of the states are known, it is easy to construct the eigenvectors themselves. Normalization of the eigenvectors combined with the conditions on the components contained in Eq. (12) gives $x_1^{(i)}$

$$1 = |x_1^{(i)}|^2 + |x_2^{(i)}|^2 + \cdots + |x_{2n+2}^{(i)}|^2$$

$$= |x_1^{(i)}|^2 + \hbar^2\Omega^2\left\{ \frac{|x_1^{(i)}|^2}{(E_i+n\hbar\Delta)^2} + \frac{|x_1^{(i)}|^2}{(E_i+(n-1)\hbar\Delta)^2} + \cdots + \frac{|x_1^{(i)}|^2}{(E_i-n\hbar\Delta)^2} \right\},$$

or

$$|x_1^{(i)}|^2=\left[1+\hbar^2\Omega^2\left\{\frac{1}{(E_i+n\hbar\Delta)^2}+\frac{1}{(E_i+(n-1)\hbar\Delta)^2}+\cdots+\frac{1}{(E_i-n\hbar\Delta)^2}\right\}\right]^{-1}$$
$$=\left[1+\frac{\Omega^2}{\Delta^2}\left\{\frac{1}{(y+n)^2}+\frac{1}{(y+n-1)^2}+\cdots+\frac{1}{(y-n)^2}\right\}\right]^{-1}. \tag{17}$$

The series in Eq. (17) also has a simple expression in the large n limit,[10] so that Eq. (17) becomes

$$|x_1^{(i)}|^2=[1+(\Omega^2\pi^2/\Delta^2)\csc^2(E_i\pi/\hbar\Delta)]^{-1}$$
$$=[1+(\Omega^2\pi^2/\Delta^2)(1+\cot^2(E_i\pi/\hbar\Delta))]^{-1}$$
$$=\left[1+\frac{\Omega^2\pi^2}{\Delta^2}+\frac{E_i^2}{\hbar^2\Omega^2}\right]^{-1}, \tag{18}$$

where we have used the characteristic equation, Eq. (16), to simplify the final result.

The coefficients a_i in Eq. (10) are equal to the components $x_1^{(i)}$, so the amplitude for the system to be found in its initial state with the atom excited is

$$\langle e;\varnothing|\psi(t)\rangle=\sum_i\left[1+\frac{\Omega^2\pi^2}{\Delta^2}+\frac{E_i^2}{\hbar^2\Omega^2}\right]^{-1}\exp(-iE_it/\hbar)$$
$$=\sum_{E_i>0}2\left[1+\frac{\Omega^2\pi^2}{\Delta^2}+\frac{E_i^2}{\hbar^2\Omega^2}\right]^{-1}\cos(E_it/\hbar). \tag{19}$$

In arriving at the final line of Eq. (19), we have used the fact that each energy E_i is matched with an energy of equal magnitude and opposite sign. This is clear because the characteristic equation is odd in E_i. The sum in the final line is only over positive eigenvalues.

The expression in Eq. (19) resembles a conventional Fourier series in that it is a sum over cosine functions, but the arguments of the cosine are not equally spaced. They are, rather, proportional to the roots of the transcendental characteristic equation. The behavior of this expression as a function of time is not obvious. The numerical evaluation of this equation is easy, however, once the eigenvalues are determined. Upon plotting this function two remarkable features become immediately apparent, as illustrated in Fig. 1. First, the amplitude for the system to remain in the state $|e;\varnothing\rangle$ with the atom excited exhibits exponential decay for $t<2\pi/\Delta$:

$$\langle e;\varnothing|\psi(t)\rangle=\exp(-\gamma t), \tag{20}$$

where the decay constant is given by

$$\gamma=\frac{\pi\Omega^2}{\Delta}. \tag{21}$$

The decay constant may appear to depend on the length of the cavity because of the length dependence of Δ. The factor Ω^2 is proportional to $1/V$, though, and the length dependence cancels. The second feature of the time evolution is that the amplitude for finding the atom in its excited state undergoes

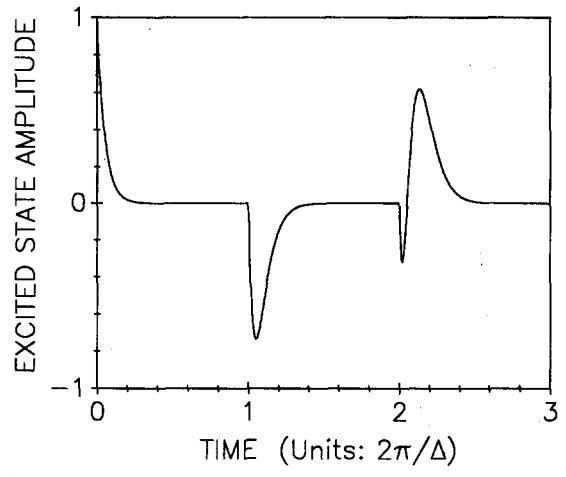

Fig. 1. Single atom at center of multimode cavity: Amplitude for the system to be found in the state $|e;\varnothing\rangle$ (excited atom and no photon) as a function of time. This is a plot of Eq. (19) in the text. Note the initial exponential decay and the subsequent revivals. The atom-field coupling strength, Ω, for this plot is set equal to the mode frequency spacing, Δ.

abrupt recurrences. The onsets of these recurrences are spaced by a regular time interval,

$$T_{\text{recurrence}}=2\pi/\Delta. \tag{22}$$

It is not easy to see that Eq. (19) should yield exponential decay, but Stey and Gibberd[9] have demonstrated by different techniques that Eq. (20) is indeed the exact solution for the amplitude in the interval $0<t<2\pi/\Delta$. In the next interval, $2\pi/\Delta<t<4\pi/\Delta$, Stey and Gibberd have expressed the amplitude as

$$\langle e;\varnothing|\psi(t)\rangle=\exp\left(-\frac{\pi\Omega^2}{\Delta}t\right)-\frac{2\pi\Omega^2(t-2\pi/\Delta)}{\Delta}\times\exp\left\{-\frac{\pi\Omega^2}{\Delta}(t-2\pi/\Delta)\right\}$$
$$\text{for } 2\pi/\Delta<t<4\pi/\Delta, \tag{23}$$

which displays the first recurrence of the atomic excitation in the second term. After each interval of $2\pi/\Delta$, they show that an additional term is added on to the amplitude which is in the form of a Laguerre polynomial times an exponentially decaying factor. This is exactly the behavior that arises from the sum in our Eq. (19), which is valid as written for all times.

The recurrences are easy to interpret in the context of quantum optics. As mentioned above, for an atom at the center of a cavity of length L, the effective spacing in angular frequency between the modes with which the atom interacts is $2\pi c/L$. Thus, the recurrences in the atomic excitation occur every L/c, which is simply the time it takes the "photon" to propagate to the ends of the cavity and return, with the resultant probability of re-exciting the atom. The speed of light is built into the problem via the electromagnetic mode spacing in the cavity.

Conventional treatments of spontaneous emission using the Wigner–Weisskopf Hamiltonian[7-8] result in the same exponential decay that we obtain, although all information about the recurrences of the atomic excitation is lost in these treatments due to the continuum limit for the field spectrum and the approximations that make it possible to

solve the integrodifferential equation for the amplitude of the state $|e;\varnothing\rangle$. More sophisticated treatments of spontaneous emission that do not include the approximations inherent in the Wigner–Weisskopf model are known to give small deviations from exact exponential decay at very long and very short times.[11–15] These features do not appear in our solution. We find it somewhat surprising that in the exact solution of the Wigner–Weisskopf Hamiltonian, which retains information about recurrences at long times, the decay maintains its exact exponential form during the finite interval $0<t<2\pi/\Delta$.

The preceding calculations assumed that the atom was stationary at the center of the cavity. For atoms which are not at the center of the cavity, the problem is only slightly more complicated. For example, when the atom is at a position $L/4$ from one end of the cavity, the coupling strengths to the modes with which the atom interacts are not equivalent, due to the sinusoidal spatial dependence of the radiation mode. The effective coupling constants are

$$\Omega_n=\Omega\,\sin(n\pi/4)=\pm\Omega,\pm\frac{\Omega}{\sqrt{2}},\quad \text{or } 0. \tag{24}$$

Thus, the top row and left column of the matrix of Eq. (6), which contain the coupling constants, must be modified. The new matrix to be diagonalized is

$$\begin{pmatrix} 0 & \hbar\Omega & \hbar\Omega/\sqrt{2} & 0 & \cdots \\ \hbar\Omega & -n\hbar\Delta' & 0 & 0 & \\ \hbar\Omega/\sqrt{2} & 0 & -(n-1)\hbar\Delta' & 0 & \\ 0 & 0 & 0 & -(n-2)\hbar\Delta' & \\ & \vdots & & & \ddots \\ & & & & n\hbar\Delta' \end{pmatrix}. \tag{25}$$

In Eq. (25), $\Delta'=2\pi(c/2L)$, whereas before the effective mode spacing was $2\pi(c/L)$.

The mathematics follows exactly the same lines as before, and the resulting characteristic equation is

$$\pi\left\{\cot\left(\frac{\pi y}{4}\right)+\frac{1}{2}\cot\left[\frac{\pi}{4}(y+1)\right]+\frac{1}{2}\cot\left[\frac{\pi}{4}(y-1)\right]\right\} =4\left(\frac{\Delta'}{\Omega}\right)^2 y. \tag{26}$$

The time-dependent amplitude to find the atom in the excited state is again given by

$$\langle e;\varnothing|\psi(t)\rangle=2\sum_{E_i>0}|x_1^{(i)}|^2\cos(E_i t/\hbar), \tag{27}$$

where now the first components of the eigenvectors are given by

$$|x_1^{(i)}|^2=\left[1+\frac{\pi^2}{32}\left(\frac{\Omega}{\Delta'}\right)^2\left\{4+2\cot^2\left[\frac{\pi}{4}y\right] +\cot^2\left[\frac{\pi}{4}(y+1)\right]+\cot^2\left[\frac{\pi}{4}(y-1)\right]\right\}\right]^{-1}. \tag{28}$$

Figure 2 illustrates a plot of Eq. (26) for the case of the atom at $L/4$ from the end. The atom decays as before, but the amplitude for the atom to be in the excited state exhibits recurrences spaced by one-half of the previous value, or

$$T'_{\text{recurrence}}=2\pi/4\Delta'=L/2c. \tag{29}$$

This is just the time it takes for the light to travel to the nearest end wall of the cavity and back, and then be reabsorbed by the atom.

Interaction of a bound quantum system with a continuum is often said to result in irreversible decays of quantum systems. In this example, an atom interacts with a quasicontinuum and exhibits revivals in the amplitude for excitation. The revival in the excitation will not be realized in practice in a system like this if the mode spacing is so small as to push the recurrence time too far into the future. If the recurrence time is too long, the interaction of the atom +quasicontinuum system with the outside world will certainly destroy any chance to observe such recurrences.

The phenomenon of decay and revival of the atomic excitation in this model is very reminiscent in some respects of the periodic spontaneous collapses and revivals predicted for Rabi oscillations in the coherent state Jaynes–Cummings model discovered by Eberly *et al.*[16] They examined a single two level atom interacting with a single mode of a radiation field prepared in a coherent state, i.e., the most nearly classical of quantum radiation states. In a single photon number state, the Rabi frequency characterizes the time scale of the atom's oscillation between the ground and excited states under the influence of the single radiation mode. If the field is in a coherent state, the time evolution of the system involves a sum over many sinusoidal Rabi oscillations because of the distribution of the photon number states which make up the coherent state field. The oscillations begin in phase, but as they dephase, the observable Rabi oscillations of the atom die out. Eberly *et al.*[16] pointed out that the oscillations induced by the various number states must eventually, at least partially, rephase. This occurs because only a finite number of terms contribute significantly to the sum, and thus they will rephase in a time characterized by their frequency separation.

Collapses and revivals due to dephasing and rephasing of quantum states have been observed experimentally in a system consisting of an atom interacting with the modes of a micromaser cavity.[17] The phenomenon is a general one, and has also been observed in the rephasing of spatially localized wave packets in Rydberg atoms.[18] In the case of the Rydberg atoms, it is the discrete spacing of the Rydberg levels that leads to the revivals. In the model presented in this paper, the eigenvalues of the total Hamiltonian are approximately sepa-

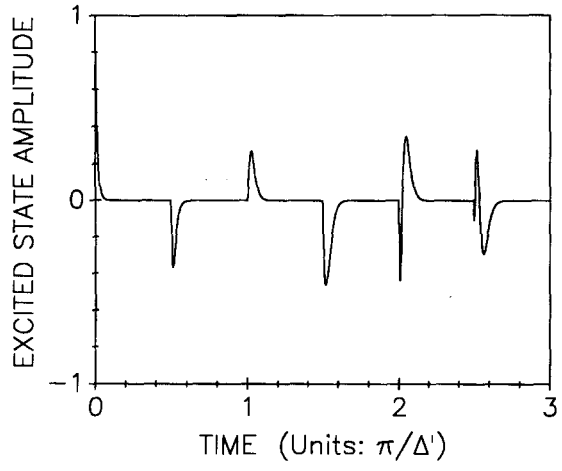

Fig. 2. Single atom in multimode cavity at position $L/4$ from nearest end wall: Amplitude for the system to be found in the state $|e;\varnothing\rangle$ (excited atom and no photon) as a function of time. This is a plot of Eq. (27) in the text. The atom initially exhibits exponential decay, but the subsequent revivals occur with half the period of those illustrated in Fig. 1. The atom-field coupling strength, Ω, for this plot is set equal to twice the mode frequency spacing, Δ.

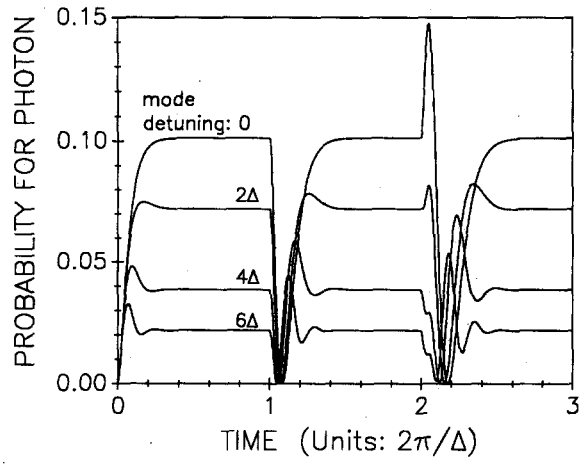

Fig. 3. Single atom in center of multimode cavity: Probabilities for the system to be in representative states $|g;n\rangle$ (ground state atom, one photon in mode with detuning $n\Delta$) as a function of time. The probabilities initially grow with time and then reach a relatively constant value until the first revival of the atomic excitation at time $T_{\text{recurrence}}=2\pi/\Delta$. Atomic parameters are the same as in Fig. 1.

rated by the intermode spacing Δ, and only a finite number of them contribute appreciably to the sum in Eq. (19) which gives the time evolution of the system. The different terms in this sum will certainly dephase in time because the spacings are not exact multiples of Δ, but simple consideration of the discrete nature of the spectrum leads to the expectation of rephasing at a time characterized by $1/\Delta$.

III. SPECTRUM OF AN ATOM IN A ONE-DIMENSIONAL MULTIMODE CAVITY

In order to find the spectrum of the photons emitted by the atom in the one-dimensional cavity treated in Sec. II, it is necessary to examine the amplitudes for the states which contain one photon and a ground state atom. Consider the amplitude for the mode which is detuned from the atomic resonance frequency by $m\Delta$, i.e., the amplitude for the system to be in the state $|g;m\rangle$. Using the results from Sec. II gives

$$\langle g;m|\psi(t)\rangle=\langle g;m|\sum_i \{\exp(-iE_it/\hbar)|E_i\rangle\langle E_i|e;\varnothing\rangle\}$$

$$=\sum_i x_1^{(i)}x_{n+m+2}^{(i)*}\exp(-iE_it/\hbar). \tag{30}$$

Now using Eq. (12) to rewrite $x_{n+m+2}^{(i)}$ in terms of $x_1^{(i)}$, we obtain

$$\langle g;m|\psi(t)\rangle=\pm\sum_i \frac{\hbar\Omega|x_1^{(i)}|^2}{(E-m\hbar\Delta)}\exp(-iE_it/\hbar).$$

$$=\pm\sum_i \hbar\Omega\left[\left(1+\frac{\Omega^2\pi^2}{\Delta^2}+\frac{E_i^2}{\hbar^2\Omega^2}\right)(E_i-m\hbar\Delta)\right]^{-1}$$

$$\times\exp(-iE_it/\hbar). \tag{31}$$

The sign of this amplitude is given by the sign of the coupling coefficient Ω_m.

The corresponding probabilities as a function of time for several representative photon modes are plotted in Fig. 3. After the initial decay of the atom is essentially complete, and before the first revival of the atomic excitation, the system is in a relatively stable state and none of the amplitudes change appreciably with time. At such times the spectrum of the photons emitted by the decay can be inferred from the probabilities of photons being in the various modes. These are given by the square of amplitudes like those in Eq. (31). Representative probabilities at times long compared with the atomic decay time are plotted versus mode detunings in Fig. 4. An analytic expression for the spectrum is difficult to derive exactly from Eq. (31), but the form of the spectrum is clear from the numerical calculations. The probabilities fit a Lorentzian curve with full width at half maximum exactly equal to twice the decay rate γ of Eq. (21). This gives the usual lifetime-linewidth uncertainty relationship.

In the model above, the spectrum is centered exactly on the value of the bare atomic resonance. This must be the case

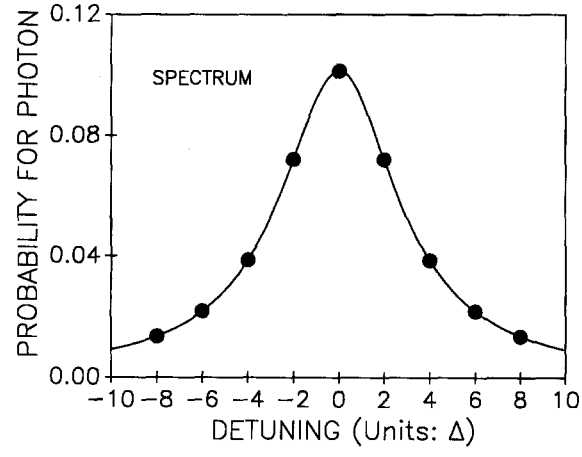

Fig. 4. Single atom at center of multimode cavity: Spectrum of photon probabilities at "long" times. The points are taken from the data plotted in Fig. 3 at time $t=6/\Delta$. This time is well after the atom has decayed, but before the first revival at time $T_{\text{recurrence}}=2\pi/\Delta$. The line is a Lorentzian curve with a full width at half-maximum equal to 2γ.

because there is perfect symmetry in the Hamiltonian between states with photons of equal and opposite detunings, e.g., between $|g;m\rangle$ and $|g;-m\rangle$. Spectral shifts will occur when the coupling to the continuum does not display this symmetry.

IV. DECAY OF AN ATOM IN A DAMPED, SINGLE-MODE, RESONANT CAVITY

Another problem involving damping that can be handled easily with the techniques employed above is that of an atom interacting with a single mode of a damped cavity. The damping of the cavity mode strongly modifies the decay of the atom from the exponential decay the atom would undergo in free space. These effects have been studied in detail by Sachdev,[19] Haroche,[20] and Knight.[21] In a high Q cavity the photon lifetime in the cavity is long enough that the atom undergoes damped Rabi oscillations, while in the low Q, or highly damped cavity, the atoms will decay exponentially, but at a rate which is enhanced from the free space decay rate by a factor proportional to Q. Such effects have been experimentally observed in open Fabry–Perot cavities at millimeter wavelengths in Rydberg atoms[22] as well as at optical frequencies.[23]

To model the effect of the damping of the cavity mode, we assume that the electromagnetic radiation within the cavity is coupled to a quasicontinuum of oscillator states in the cavity walls or mirrors. For ease of calculation, we assume that this bath of wall states consists of a set of harmonic oscillators which are equally spaced in angular frequency by an amount Δ. The coupling of the radiation mode to each of the bath states is assumed equivalent, and analogous to the rotating-wave-approximation coupling of a radiation mode to a two level atom. The atom is not directly coupled to the wall states, but only indirectly coupled through the interaction with the single mode of the electromagnetic field. The single radiation mode will be assumed to be resonant with the atomic frequency ω_0.

The Hamiltonian for the total system can be written

$$H=H_{\text{atom}}+H_{\text{field}}+H_{\text{wall}}+H_{\text{atom-field}}+H_{\text{field-wall}}, \tag{32}$$

where H_{atom} is the unperturbed two-level atomic Hamiltonian, H_{field} is the Hamiltonian for the single radiation mode, H_{wall} is the Hamiltonian for the oscillators in the walls of the cavity, and the remaining terms express the interactions between the atom and radiation, and the radiation and the walls. If c_i and $c_i^\dagger$ are the annihilation and creation operators for the modes in the walls, and Ω_1 and Ω_2 are the radiation–atom and radiation–wall coupling constants respectively, then the rotating wave approximation Hamiltonian can be written

$$H=\hbar\omega_0 b^\dagger b+\hbar\omega_0 a^\dagger a+\sum_i \hbar\omega_i c_i^\dagger c_i+\hbar\Omega_1(ab^\dagger+a^\dagger b)$$
$$+\sum_i \hbar\Omega_2(ac_i^\dagger+a^\dagger c_i). \tag{33}$$

As a set of basis states we use the following:

$|e;0;\varnothing\rangle\leftrightarrow$ e.s. atom, no photon, no wall excitation,

$|g;1;\varnothing\rangle\leftrightarrow$ g.s. atom, one photon, no wall excitation,

$|g;0;m\rangle\leftrightarrow$ g.s. atom, no photon, one wall excitation in mode with detuning $m\Delta$.

A matrix representation of the Hamiltonian of Eq. (33) is given by

$$\begin{pmatrix} 0 & \hbar\Omega_1 & 0 & 0 & \cdots & 0 \\ \hbar\Omega_1 & 0 & \hbar\Omega_2 & \hbar\Omega_2 & \cdots & \hbar\Omega_2 \\ 0 & \hbar\Omega_2 & -n\hbar\Delta & 0 & & \\ 0 & \hbar\Omega_2 & 0 & -(n-1)\hbar\Delta & & \\ \vdots & \vdots & & & \ddots & \\ 0 & \hbar\Omega_2 & & & & +n\hbar\Delta \end{pmatrix}. \tag{34}$$

The procedures applied to the atom in a multimode cavity in Sec. II are also easy to apply in this case. For a finite number of wall oscillators spanning a range of detunings from $-n\Delta$ to $n\Delta$, the matrix in Eq. (34) has $2n+3$ dimensions. Writing out the energy eigenvalue equations analogous to Eq. (12) reveals an eigenvalue of $E=0$, and in the limit of large n, the remaining eigenvalues are given by the characteristic equation

$$1/y+(\Omega_2/\Omega_1)^2\pi\cot(\pi y)=(\Delta/\Omega_1)^2 y, \tag{35}$$

where once again, $y=E_i/h\Delta$. Like Eq. (16), this equation has roots between each of the integers, which makes it easy to solve numerically. We calculate the amplitude for the atom to be in the excited state in a manner similar to that used in Sec. II and find as before

$$\langle e;0;\varnothing|\psi(t)\rangle=\sum_i |\langle e;0;\varnothing|E_i\rangle|^2\exp(-iE_i t/\hbar), \tag{36}$$

where

$$|\langle e;0;\varnothing|E_i\rangle|^2 = \begin{cases} \left[1+\left(\dfrac{E_i}{\hbar\Omega_1}\right)^2+\left(\dfrac{\Omega_2 E_i\pi}{\Omega_1\hbar\Delta}\right)^2\csc^2\left(\dfrac{E_i\pi}{\hbar\Delta}\right)\right]^{-1}, & E_i\neq 0 \\ \left[1+\left(\dfrac{\Omega_1}{\Omega_2}\right)^2\right]^{-1}, & E_i=0. \end{cases}$$

The time behavior of the amplitude is once again not transparent, but we will demonstrate that the amplitude displays either overdamped or underdamped oscillations, depending on the characteristics of the cavity.

Cavities are conventionally characterized by their Q value, or equivalently by a damping constant, γ_{cav}, which describes the loss rate of energy from the cavity. These two quantities are related by

$$Q=\omega_0/\gamma_{\text{cav}}. \tag{37}$$

To relate these quantities to the cavity parameters Ω_2 and Δ, consider the Hamiltonian of Eq. (33) in the absence of any atomic degrees of freedom. This is then exactly the Hamiltonian of Sec. II, with the radiation mode playing the role of the atomic excitation, and the cavity oscillators playing the role of the multimode radiation field. The decay rate for energy initially in the electromagnetic radiation mode is twice the decay rate for the amplitude. The decay rate for the amplitude can be read off directly from the results of our earlier calculation as expressed in Eq. (21). This gives

$$\gamma_{\text{cav}}=2\pi\Omega_2^2/\Delta, \tag{38}$$

or

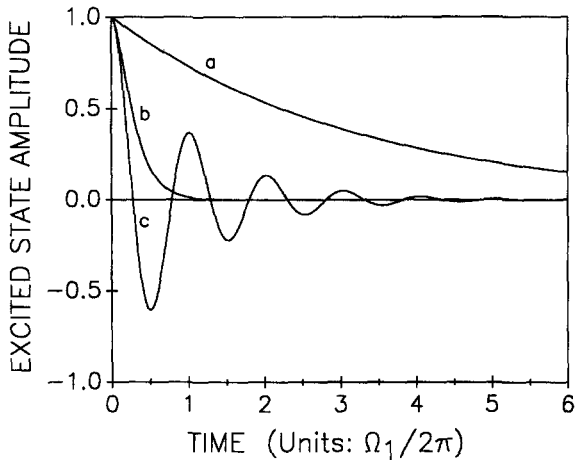

Fig. 5. Single atom in a single mode, damped cavity: Amplitude for the system to be in the state $|e;0;\varnothing\rangle$ (excited atom, no photon, no wall excitations) as a function of time. This is a plot of Eq. (36) in the text. In curve "a" the atom-field oscillations are overdamped ($\gamma_{cav}/\Omega_1=40$), in "b" the oscillations are critically damped ($\Omega_{cav}/\Omega_1=4$), and in "c" the oscillations are underdamped ($\gamma_{cav}/\Omega_1=0.63$).

$$Q=\frac{\omega_0\Delta}{2\pi\Omega_2^2}. \tag{39}$$

Figure 5 gives representative plots of the time-dependent amplitude for the atom to be in its excited state as given by Eq. (36). Plot "*a*" illustrates the case of a low Q cavity. In terms of fundamental parameters this means that the coupling of the atom to the walls of the cavity is stronger than the coupling to the radiation field, or $\gamma_{cav}\gg\Omega_1$. In this limit the decaying amplitude is fit well by the exponential

$$\langle e;0;\varnothing|\psi(t)\rangle=\exp\left(-\frac{\Delta}{\pi}\left(\frac{\Omega_1}{\Omega_2}\right)^2 t\right)=\exp\left(-\frac{2Q\Omega_1^2}{\omega_0}t\right). \tag{40}$$

The effect of the damped cavity is to alter the decay rate of the atom from the rate in free space. To compare the decay rate of Eq. (40) to the free-space decay rate of probability we need to recall the explicit form for the constant Ω_1 which characterizes the coupling between the field and the atom which is given in Eq. (3), and use it in conjunction with the expression[24]

$$\gamma_{free}=\frac{1}{4\pi\epsilon_0}\frac{4\omega^3 d^2}{3\hbar c^3}, \tag{41}$$

with the result that

$$\gamma_{\text{atom in cavity}}=\left(\frac{3Q\lambda^3}{4\pi^2 V}\right)\gamma_{free}. \tag{42}$$

In a Fabry–Perot cavity, the mode volume V is on the order of λ^3, which means that the spontaneous decay rate in this limit will be enhanced by a quantity on the order of Q of the cavity, a result which was first suggested by Purcell in 1946.[25]

In the high Q limit, or $\Omega_1\gg\gamma_{cav}$, the cavity retains photons long enough that the energy will be exchanged between the atom and the electromagnetic mode several times before the ultimate loss of the energy to the modes in the walls of the cavity. This underdamped oscillation of the amplitude for the system to be found in the state $|e;0;\varnothing\rangle$ is displayed in Fig. 5, and is well fit by

$$\langle e;0;\varnothing|\psi(t)\rangle=e^{-\gamma_{cav}t/4}\cos\left(\sqrt{\Omega_1^2-\gamma_{cav}^2/16}\,t\right). \tag{43}$$

This is consistent with the analytical results of Knight[21] and Haroche[20] which were derived using other techniques.

V. CONCLUSIONS

We have solved, exactly, two simple quantum optical models in which the system of interest is coupled to a large number of external degrees of freedom. The resulting damping of the system of interest depends on the properties of the external reservoir and the coupling to the reservoir. The understanding of damping phenomena that such models give has lead to many investigations in recent years which demonstrate that spontaneous emission, a seemingly fundamental property of an atom, can be significantly modified by altering the properties of the vacuum to which the atom is coupled. Such research has been summarized recently by Haroche and Kleppner.[26] The simple quantum mechanical methods used in this paper to solve simple models of spontaneous emission should make the fundamental ideas of such research widely accessible.

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Observing thermomigration of air bubbles in a fluid under gravity at an undergraduate research laboratory

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An educational experimental project in fluid mechanics for space science or physics honors undergraduates is presented. The motion of air bubbles in a viscous silicone oil in response to buoyancy and Marangoni convection is studied with an experimental setup adapted to the undergraduate laboratory. The Marangoni effect or thermocapillary convection occurs when a temperature gradient is present in the oil which generates a surface tension differential over the bubbles. Results of measured radii and temperature gradients necessary to balance the buoyancy and Marangoni forces, along with computed Marangoni velocities are in good agreement with previous terrestrial and space experiments. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

Finding a fluid mechanics phenomenon for an advanced undergraduate space science experimental project that can be observed both at 1 and 0 g is not obvious. The Marangoni effect or "thermocapillary convection," a phenomenon now considered critical to control the manufacturing process and production of homogeneous materials in space, was in fact observed in both environments. Materials processing and manufacturing in space under microgravity conditions is the subject of intensive research and is a burgeoning field of physics. Under microgravity, bubbles resulting from solidification during materials processing do not move because of the absence of buoyancy. However, under the same conditions, the presence of electric and magnetic fields, temperature and concentration gradients, for example, can cause bubble motion alone.[1]

The thermomigration phenomenon, which was discovered in 1871 by the Italian scientist C. G. M. Marangoni, is induced by a temperature gradient as a consequence of the temperature dependence of the surface tension which gives rise to bubble motion. Marangoni convection is in reality a very complex physical phenomenon, most notably if treated tridimensionally. Theoretical models have been attempted with reasonable success, but the true solution seems to lie in the use of computational fluid dynamics. Axisymmetric models have, however, been attempted and experimental results are in surprisingly good agreement with theory. Young *et al.*,[2] a classical paper, were the first to study the combined effects of buoyancy and thermocapillary convection by adapting the model that Hadamard[3] and Rybczynski[4] had separately developed for the motion of a droplet in a liquid due to buoyancy only (assuming creeping flow). The experimental method of Young *et al.* was quite original: bubbles were injected in a vertical liquid column which was heated from below and cooled from above in such a way that the buoyancy force and the Marangoni effect would cancel out and the bubble would become motionless. This is due mainly to the fact that the bubbles will move to the hot side to minimize their surface tension energy. In 1979, Hardy[5] perfected the procedure and was able to obtain more accurate results. Our experimental setup is based on Young's and Hardy's method but was adapted for the undergraduate laboratory and requires essentially only basic instrumentation. The following treatment of an uncommon physical phenomenon such as the Marangoni effect can be of interest to the undergraduate curriculum for those who may want to reproduce current space experiments under gravity conditions.

II. THEORY

The complete derivation of the Young model is beyond the reach of an advanced undergraduate fluid mechanics course but can be adapted for undergraduates without loss of understanding if the emphasis is put on the underlying physics. In summary, the axisymmetric model is derived from the linearized Navier–Stokes equations solved for the creeping flow approximation (i.e., for very small bubbles in a nonconvective fluid) allowing the use of Laplace's equation to define

