The calculated photon: Visualization of a quantum field

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We calculate spatial localization and phase properties of spontaneously emitted photons. Our model is simple yet fully quantized: the emitting atom is a two-level atom located in a one-dimensional multimode optical cavity. Although the photon state vector does not have a position-space representation, the expectation value of the square of the electric field operator (intensity) is spatially localized and this pattern shifts at the speed of light. The emitted photon exhibits classical-like phase properties in the intensity expectation value when it ''interferes with itself'' after reflection. The phase properties of the emitted radiation are also evident at times sufficiently long for the radiation to have returned to the emitting atom before the decay of excited atomic state is complete.
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I. INTRODUCTION

The dual wave–particle nature of quantum objects is discussed in almost all introductory texts on modern physics and quantum mechanics, but it is not always clear which aspects of classical wave behavior are retained in a fully quantum-mechanical treatment, or where to draw the line between wave-like aspects and particle-like aspects and how to justify the division. Photons are perhaps the simplest objects that can be treated with a quantum field theory that makes the distinctions clear. In this paper we present a simple yet fully quantized model of spontaneous emission and use it to illustrate graphically some of the properties of the emitted electromagnetic field. Speaking loosely, we answer the question ''What does a photon look like?'' and provide a quantitative complement to discussions of the concept of the photon like those found in Refs. 1–3.

A conceptual understanding of quantized electromagnetic radiation, or photons, is complicated by (at least) two factors. First, the quantum state vectors that represent photons do not have position-space representations that are analogous to the familiar wave functions of nonrelativistic quantum mechanics.[4] Thus there is no quantity analogous to $|\psi|^2$ giving a probability density for finding the photon in space. Second, the definition of a quantum operator corresponding to the classical phase of an electromagnetic wave has been problematic.[5] We sidestep these somewhat tricky issues by investigating other quantum mechanical quantities that give information related to the classical concepts of phase and spatial localization.

We consider the field spontaneously emitted by a single two-level atom with a fixed position in a one-dimensional multi-mode optical cavity. The atom is coupled to all of the modes of the cavity, and due to this coupling an atom that is initially in the excited state can potentially emit a photon into any of the cavity modes. More precisely, the state of the system evolves from an excited atom with no photons to a linear combination of the initial state and all the one-photon states. The fact that the atom is located within a cavity simplifies the calculations, and the cavity walls also produce reflections that cause the radiation to return to the emitting atom, in some sense interfering with itself along the way.

In order to see the spatial localization of the photon, we calculate the expectation value of the square of the electric field operator, which is proportional to the energy density of the field. (See Ref. 4 for a more complete discussion of the problem of photon localization.) The results are displayed graphically in the figures of Sec. IV. The energy density expectation has a spatial envelope that accords with classical intuition appropriate to radiation emitted by a classical oscillating damped dipole. This envelope travels with a speed c as expected. However, unlike a classical traveling wave, there is no sinusoidal modulation of the energy density at the classical wavelength as the radiation pulse travels away from the atom. Thus there is no obvious analog to the classical phase associated with the radiation.

Phase properties of the emitted radiation reveal themselves after reflection from the cavity walls. At times after radiation could have reached the end walls, the expectation value of intensity reveals a pattern that can be interpreted as outgoing and returning pulses, and in the region of overlap the radiation ''interferes with itself,'' producing a standing wave pattern appropriate for the classical resonance wavelength. The interaction of the reflected radiation with the emitting atom also exhibits classical-like phase effects: the effect of the radiation on the quantum *amplitude* for the atom to be in the excited state depends critically on the distance traveled by the radiation.

Many of the properties of the emitted photon have close analogs in the classical radiation emitted by an oscillating dipole, and many of the phenomena of cavity quantum electrodynamics can be understood in terms of these quasiclassical properties. Dowling[6] has pointed out the classical nature of spontaneous emission in cavities, and our work provides a fully quantum mechanical counterpart to his classical calculations.

In Sec. II we describe the mathematical model we use for the emitting atom in a cavity, and in Sec. III we derive the expressions we use for the observable properties of our system. The graphical results are displayed in Sec. IV. The points that we make about the nature of spontaneously emitted photons can be understood by reference to the graphs in Sec. IV alone.

II. PHYSICAL SYSTEM: HAMILTONIAN, BASIS STATES, AND OPERATORS

We consider the problem of a single two-level atom in a one-dimensional multi-mode optical cavity. The ground and excited states of the atom will be denoted $|g\rangle$ and $|e\rangle$, re-

spectively, and the zero-field energy separation between atomic states is $\hbar\omega_{eg}$. The cavity extends from $x=0$ to $x=L$ with perfectly reflecting mirrors at the ends, and the atom is fixed at position x_a. The classical electromagnetic modes of the system have spatial dependence

$$\mathcal{E}(x)\sim\sin\left(\frac{n\pi x}{L}\right), \tag{1}$$

where n is a positive integer. The angular frequencies of the modes are given by

$$\omega_n=\frac{n\pi c}{L}, \tag{2}$$

and the modes are equally spaced with angular frequency separation

$$\Delta_c=\frac{\pi c}{L}. \tag{3}$$

Some radiation modes will closely match the resonant frequency of the two-level atom, and others will not. It is not necessary for any mode to match the atomic resonance exactly, but it is convenient to enumerate modes from the even-numbered mode that is closest to the atomic resonance. We designate the mode number for this near-resonant mode as j_0, and label the frequency of this mode as ω_0. The detuning of this mode from atomic resonance ω_{eg} is δ, defined such that

$$\omega_0=\omega_{eg}+\delta. \tag{4}$$

Mode numbers for general modes will then be written in terms of their separation from the most nearly resonant mode as $n=j_0+j$, where j is an integer. Throughout this paper we assume that the resonant frequency is much larger than the cavity mode spacing, that is, $\omega_{eg}\gg\Delta_c$. We do not, however, assume anything about the relative magnitude of Δ_c and the free-space natural linewidth of the atom.

The quantization of the free electromagnetic field is covered in many standard references (see, for example, Refs. 4, 7–9) and has been reviewed in a recent article in this journal.[10] The modes of the cavity are treated as independent quantized harmonic oscillators, and the eigenstates of the free field are denoted by the occupation numbers of the individual modes: $|n_1,n_2,n_3,\ldots\rangle$. In this paper we will be concerned with situations in which there are either 1 or 0 photons in the field, so that either all occupation numbers are 0, or all but one of the occupation numbers is 0. We will condense the notation in the following manner.

- $|0,0,0,\ldots\rangle\rightarrow|\varnothing\rangle$: No photons in the field ($n_i=0$ for all i).
- $|0,0,\ldots 0,n_j=1,0,\ldots\rangle\rightarrow|j\rangle$: One photon in mode j ($n_j=1$, $n_i=0$ for $i\neq j$).

The free-field Hamiltonian is written in terms of the lowering and raising operators a_j and $a_j^\dagger$ for each mode, and is equivalent to a sum of independent harmonic oscillator Hamiltonians:

$$H_{\text{field}}=\sum_j\hbar\omega_j\left(a_j^\dagger a_j+\frac{1}{2}\right). \tag{5}$$

(The photon number operator is simply H_{field} without the $\frac{1}{2}$ terms.)

The basis states of the combined atom-field system are given by the direct product of the atom states and the field states:

- $|e;\varnothing\rangle$: Atom in the excited state, no photons in the radiation field.
- $|g;j\rangle$: Atom in the ground state, one photon in the radiation field with a detuning from the even-numbered mode closest to resonance given by $\omega_j-\omega_0=\omega_j-(\omega_{eg}+\delta)=j\Delta_c$. (The detuning of the photon from the zero-field atomic resonance frequency is $\omega_j-\omega_{eg}=j\Delta_c+\delta$.)

The electric field operator[7] is a sum over terms for each mode, each term exhibiting a sinusoidal spatial dependence identical to the classical mode structure:

$$\mathcal{E}(x)=\sum_j C_j(a_j+a_j^\dagger)\sin\left[(j_0+j)\frac{\pi x}{L}\right], \tag{6}$$

where

$$C_j=\left(\frac{\hbar\omega_j}{\epsilon_0 V}\right)^{1/2}, \tag{7}$$

with V the effective volume of the mode,[11] and ϵ_0 the permittivity of free space. The modes are enumerated as before from the even-numbered mode with the frequency closest to the zero-field atomic resonance. As discussed above, we will be assuming that the atomic resonance frequency ω_{eg} will be much greater than the mode spacing in this problem, so that the factors C_j will be not vary significantly for all modes of interest in the problem, and we will assume that this factor is given by the constant $C=\sqrt{\hbar\omega_{eg}/(\epsilon_0 V)}$ for all modes.

Classically, the energy density of the field is proportional to $\mathcal{E}^2$. The relevant quantum mechanical analog is the expectation value of the operator

$$\mathcal{E}(x)^2=\sum_{j,j'} C^2\sin\left[(j_0+j)\frac{\pi x}{L}\right]\sin\left[(j_0+j')\frac{\pi x}{L}\right]\times(a_j a_{j'}+2a_j^\dagger a_{j'}+a_j^\dagger a_{j'}^\dagger,+\delta_{j,j'}). \tag{8}$$

If we consider a single-mode field state with definite occupation number, the expectation value of the electric field operator, $\langle\mathcal{E}\rangle$, is zero. But the classical spatial mode structure of such a state is revealed in the expectation value of the square of the field operator, $\langle\mathcal{E}^2\rangle$.

In the electric-dipole approximation the interaction Hamiltonian is given by

$$H_{\text{interaction}}=-e\mathcal{E}\cdot\mathbf{r}=\mathcal{E}\cdot\mathbf{d}, \tag{9}$$

where $\mathbf{d}$ is the electric dipole operator. The dipole operator connecting the ground and excited states can be written

$$d=\hbar(d_{eg}\sigma_+ + d_{eg}^*\sigma_-), \tag{10}$$

where d_{eg} is the dipole matrix element between the ground and excited states, and the pseudo-spin operators, which act as effective raising and lowering operators on the internal degrees of freedom of the atom, are defined as follows:

$$\sigma_+=|e\rangle\langle g|, \tag{11a}$$

$$\sigma_-=|g\rangle\langle e|, \tag{11b}$$

$$\sigma_3=|e\rangle\langle e|. \tag{11c}$$

Writing the electric field operator in terms of raising and lowering operators and the dipole operator in terms of the pseudo-spin operators gives

$$H_{\text{interaction}}=\left\{\sum_j C(a_j+a_j^\dagger)\sin\left[(j_0+j)\frac{\pi x_a}{L}\right]\right\}\times(d_{eg}\sigma_+ + d_{eg}^*\sigma_-)$$
$$\simeq\sum_j \hbar(\Omega\sigma_+ a_j+\Omega^*\sigma_- a_j^\dagger)\sin\left[(j_0+j)\frac{\pi x_a}{L}\right], \tag{12}$$

where in the last line we have made the standard rotating-wave approximation by neglecting terms with $\sigma_+ a_j^\dagger$ and $\sigma_- a_j$,[7–9] and we have combined constants into

$$\Omega\equiv d_{eg}\left(\frac{\omega_{eg}}{2\hbar\epsilon_0 V}\right)^{1/2}. \tag{13}$$

With this interaction Hamiltonian the zero and one-photon states $|e;\varnothing\rangle$ and $|g;j\rangle$ form a complete set of states. This interaction Hamiltonian is a multimode version of the standard interaction considered in many quantum optics texts.[7–9] The terms in the Hamiltonian correspond either to the absorption of a cavity photon and the transition of the atom from the ground state to the excited state, or the converse. Note that the photons do not have to be exactly resonant with the atomic transition to be absorbed by the atom. States like $|g;j\rangle$ and $|e;\varnothing\rangle$ are not eigenstates of the total Hamiltonian. Therefore, if the system starts in the state with an excited atom and no photons ($|e;\varnothing\rangle$), it is not in a state of definite energy, and absolute energy conservation should not be expected.

The total Hamiltonian is

$$H=H_{\text{atom}}+H_{\text{field}}+H_{\text{interaction}}$$
$$=\hbar\omega_{eg}\sigma_3+\sum_j \hbar\omega_j\left(a_j^\dagger a_j+\frac{1}{2}\right)+\sum_j \hbar(\Omega\sigma_+ a_j + \Omega^*\sigma_- a_j^\dagger)\sin\left[(j_0+j)\frac{\pi x_a}{L}\right]. \tag{14}$$

The sum over j includes all the modes of the cavity, and for $\omega_{eg}\gg\Delta_c$, this sum over j will be effectively a sum of an infinite number of terms. This Hamiltonian is the same as that used by Wigner and Weisskopf in their pioneering work on spontaneous emission[12] with the modification that the atom-radiation coupling is mode specific, and depends on the position of the atom relative to the spatial structure of the classical mode.

III. CALCULATION OF OBSERVABLES

In what follows the initial state of the system will be a state with an excited atom and no photons, that is,

$$|\psi(0)\rangle=|e;\varnothing\rangle. \tag{15}$$

Because the basis states with well defined photon numbers and atomic energies are *not* eigenstates of the total Hamiltonian, the state of the system evolves into a linear combination of the basis states, which can be written as

$$|\psi(t)\rangle=c_e(t)|e;\varnothing\rangle+\sum_j c_{g,j}(t)|g;j\rangle. \tag{16}$$

Our goal is to find physical quantities such as:

(1) the time dependent amplitude for the atom to be found in the excited state with no field modes excited, $c_e(t)=\langle e;\varnothing|\psi(t)\rangle$;
(2) the time dependent amplitudes for a photon to be found in one of the cavity modes, $c_{g,j}(t)=\langle g;j|\psi(t)\rangle$; and
(3) the time- and space-dependent expectation value for the square of the electric field operator, $\langle\psi(t)|\mathcal{E}^2(x)|\psi(t)\rangle$.

It is the last of these quantities that will give the best picture of the localized intensity of the photon and that will give some insight into the classical phase information that is preserved in the quantum domain.

One straightforward way to solve this problem is to find the eigenstates of the total Hamiltonian and their energies and then project the initial state of the system onto the eigenstates. The qth eigenstate of the system with energy $E^{(q)}$ can be written as the linear combination

$$|E^{(q)}\rangle=d_e^{(q)}|e;\varnothing\rangle+\sum_j d_{g,j}^{(q)}|g;j\rangle, \tag{17}$$

where $d_e^{(q)}$ and $d_{g,j}^{(q)}$ are time-independent constants chosen such that

$$H_{\text{total}}|E^{(q)}\rangle=E^{(q)}|E^{(q)}\rangle. \tag{18}$$

The determination of the eigenvalues and eigenvectors is a straightforward numerical problem; a technique for rapid calculation of large numbers of eigenvalues and eigenvectors has been presented in Ref. 13. It is also easy to determine eigenvectors and eigenvalues numerically with computational packages such as MAPLE or MATHEMATICA. Details on such calculations are reviewed in the Appendix. From this point on, it will be assumed that the energies $E^{(q)}$ and the coefficients $d_e^{(q)}$ and $d_{g,j}^{(q)}$ are known numerically. [It also is possible to use Laplace transform techniques to solve the time-dependent Schrödinger equation directly for the coefficients $c_e(t)$ and $c_{g,j}(t)$ that appear in Eq. (16). See, for example, Ref. 14.]

To determine the time evolution of the system that starts in a state $|\psi(0)\rangle$, the initial state is projected onto the eigenstates of the total Hamiltonian

$$|\psi(0)\rangle=|e;\varnothing\rangle=\sum_q |E_q\rangle\langle E_q|\psi(0)\rangle, \tag{19}$$

and because the time evolution of the energy eigenstates is known, so is the evolution of the state of interest:

$$|\psi(t)\rangle=\sum_q e^{-iE_q t/\hbar}|E_q\rangle\langle E_q|e;\varnothing\rangle. \tag{20}$$

If we expand $|E_q\rangle$ as in Eq. (17), and use $\langle e;\varnothing|E_q\rangle=d_e^{(q)}$ and $\langle g;j|E_q\rangle=d_{g,j}^{(q)}$, we find

$$|\psi(t)\rangle=\sum_q e^{-iE_q t/\hbar}\left(d_e^{(q)}|e;\varnothing\rangle+\sum_j d_{g,j}^{(q)}|g;j\rangle\right)d_e^{(q)*}$$
$$=\left(\sum_q e^{-iE_q t/\hbar}|d_e^{(q)}|^2\right)|e;\varnothing\rangle+\left(\sum_{j,q} e^{-iE_q t/\hbar}d_{g,j}^{(q)}d_e^{(q)*}\right)|g;j\rangle, \tag{21}$$

and the coefficients of the basis vectors give the time-dependent amplitudes

$$c_e(t)=\langle e;\varnothing|\psi(t)\rangle=\sum_q e^{-iE_q t/\hbar}|d_e^{(q)}|^2 \tag{22}$$

and

$$c_{g,j}(t)=\langle g;j|\psi(t)\rangle=\sum_q d_{g,j}^{(q)}d_e^{(q)*}e^{-iE_q t/\hbar}. \tag{23}$$

The energy density of the field is proportional to the expectation value of $\mathcal{E}(x)^2$ that was given explicitly in Eq. (8). It is a straightforward exercise to use this expression for $\mathcal{E}(x)^2$ along with Eq. (21) to calculate an expression for $\langle\mathcal{E}(x)^2\rangle$. This exercise involves a sum over six indices, but the fact that each of the operator terms only involves a product of two raising or lowering operators simplifies the calculation considerably. The resulting expectation value is

$$\begin{aligned}&\langle\psi(t)|\mathcal{E}^2(x)|\psi(t)\rangle\\&=2\left|\sum_{j,q} Cd_e^{(q)*}d_{g,j}^{(q)}e^{-iE_q t/\hbar}\sin\left[(j_0+j)\frac{\pi x}{L}\right]\right|^2\\&\quad+\sum_j C^2\sin^2\left[(j_0+j)\frac{\pi x}{L}\right].\end{aligned} \tag{24}$$

The last sum is just the vacuum expectation value of $\mathcal{E}^2$. The contribution of this term to the total field energy in the cavity is infinite and, as is customary, this term will be subtracted from all field expectation values in the remainder of this paper. Equivalently, the $\mathcal{E}^2$ operator could be written in normal ordered form from the start. Thus,

$$\begin{aligned}&\langle\psi(t)|\mathcal{E}^2(x)|\psi(t)\rangle\\&=2C^2\left|\sum_{j,q} d_e^{(q)*}d_{g,j}^{(q)}e^{-iE_q t/\hbar}\sin\left[(j_0+j)\frac{\pi x}{L}\right]\right|^2\\&=2C^2\left|\sum_k \langle g;k|\psi(t)\rangle\sin\left[(j_0+j)\frac{\pi x}{L}\right]\right|^2,\end{aligned} \tag{25}$$

where in the last line we have used Eq. (23). Each term in this sum is the amplitude for a photon with detuning $j\Delta$ times the spatial mode function for that photon.

IV. RESULTS

The principal results of this paper are presented in Figs. 1–6 in which we plot the time-dependent amplitude for the basis state of the system $c_e(t)$, and the expectation value for the square of electric field operator. In all cases the system starts at $t=0$ in the state $|e;\varnothing\rangle$.

In the calculations illustrated in Figs. 1–6 we use units in which

$$\frac{L}{c}=1, \tag{26}$$

which means that the time for a photon to travel the length of the cavity is 1. This choice of units implies that the spacing between cavity modes for all subsequent calculations will be

$$\Delta_c=\frac{\pi c}{L}\rightarrow\pi. \tag{27}$$

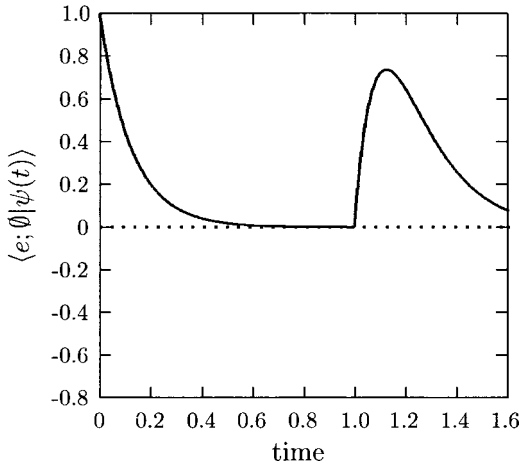

Fig. 1. The amplitude $\langle e;\varnothing|\psi(t)\rangle$ for an initially excited atom to remain in the excited state, with no photons in the field. The amplitude for the atom to be found in the excited state rises abruptly at time $t=1$ corresponding to the time at which the photon returns to the atom after reflection from the end of the cavity. (The atomic resonance frequency matches exactly the frequency of an even-numbered mode of the cavity, and the atom-field coupling for this decay is $\Omega=4$.)

The simplest case to treat is an atom at the center of the cavity, that is, $x_a=L/2$. All odd-numbered modes have antinodes at this position, so the atom–field coupling has the same magnitude for all odd modes, although the sign of the coupling alternates. Because the atom is located at a node for all even-numbered modes, it does not couple to these modes. (The decay of the atom and spontaneous emission of the photon is due to the coupling of the atom to a large number of cavity modes, and the atom might not actually couple to the mode nearest resonance.)

The results for an atom whose resonance frequency exactly matches that of an even-numbered mode ($\delta=0$) are presented in Figs. 1–3. (Notes on the details of the calculation are presented in the Appendix.) The amplitude for the atom to remain in the excited state is plotted in Fig. 1. The atom exhibits exact exponential decay until $t=L/c=1$, at which time the "photon" returns to the atom after reflection

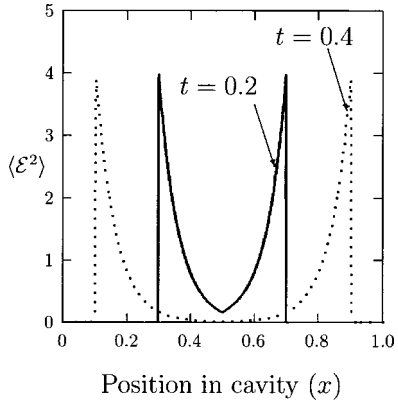

Fig. 2. The expectation value $\langle\mathcal{E}^2\rangle$ of the square of the electric field operator at time $t=0.2$ (solid line) and $t=0.4$ (dotted line). The exponential spatial dependence of the intensity reflects the exponential temporal dependence of the atomic decay. (The atom–field coupling for this decay is $\Omega=4$.)

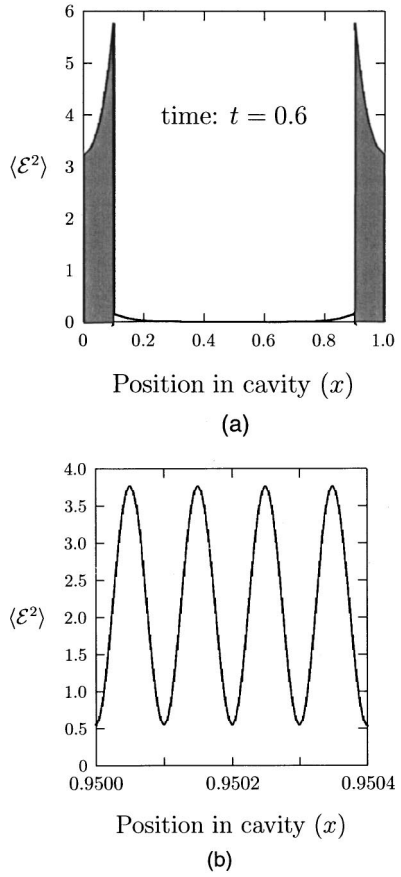

Fig. 3. The expectation value $\langle \mathcal{E}^2 \rangle$ of the square of the electric field operator at time $t=0.6$ shortly after the leading edge of the intensity expectation has hit the end wall and has reflected back on itself. The expectation value exhibits a standing wave pattern in the region where the classical outgoing and returning pulses would overlap. The region of overlap ($0.9<x<1.0$) is shaded in (a), and the details of the standing wave are illustrated in (b), which covers a much smaller spatial range within the overlap region. The minima of the standing wave pattern do not go to zero, and the standing wave ratio is exactly that which would be calculated for two classical interfering waves whose amplitude gives the exponential spatial pattern illustrated in Fig. 2.

off the ends of the cavity. For $t<L/c$, the amplitude to find an atom in the excited state with no photons in the cavity is exactly[14]

$$c_e(t)=\langle e;\varnothing|\psi(t)\rangle=e^{-\gamma t}, \tag{28}$$

where the decay constant is

$$\gamma=\frac{\pi|\Omega|^2}{2\Delta_c}. \tag{29}$$

We find it remarkable that the single sum of Eq. (22) gives the exact exponential behavior at early times, and also gives the abrupt change in atomic excitation as the radiation returns to the atom. (The re-excitation due to the returning radiation will be discussed further below.)

The expectation value of the square of the electric field is illustrated at $t=0.2$ and $t=0.4$ in Fig. 2. (All plots of the expectation value of $\mathcal{E}^2$ were made for $j_0=10\,000$. The exact value for j_0 does not affect any of the qualitative features of the graphs; it affects only the spatial scale of the interference effects that will be discussed below.) The intensity of the field has an abrupt leading edge traveling to the right and

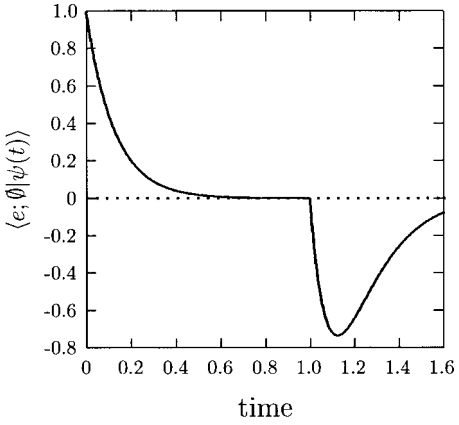

Fig. 4. As in Fig. 1, the amplitude $\langle e;\varnothing|\psi(t)\rangle$ for an initially excited atom to remain in the excited state, with no photons in the field, but in this case the atomic resonance frequency matches the frequency of an odd-numbered cavity mode of the cavity, and the amplitude for the atom to return to the excited state at time $t=1$ due to the reflection is negative. (The atom–field coupling for this decay is $\Omega=4$.)

the left, and in both directions the intensity tails off exponentially in space, which reflects the exponential decay in time of the atom. (The sharpness of leading edge is an artifact of the somewhat artificial initial condition that at $t=0$ the state of the system is purely $|e;\varnothing\rangle$.) The expectation value of intensity exhibits no spatial modulation other than the exponential envelope; there is no apparent phase information, in contrast to the sinusoidal variation of the classical traveling wave emitted by an oscillating dipole. While the expectation value of the intensity is simultaneously nonzero to the right and the left of the atom, the expectation value for the number of photons in the field is still always less than 1, and a photon could not simultaneously be detected on both sides of the atom.

At times greater than $t=0.5$, the radiation is reflected from the ends of the cavity. For such times, the wave can ''interfere with itself,'' and the expectation value of intensity exhibits spatial modulation in the region of overlap between the outgoing pulse and its reflection. This effect is illustrated in

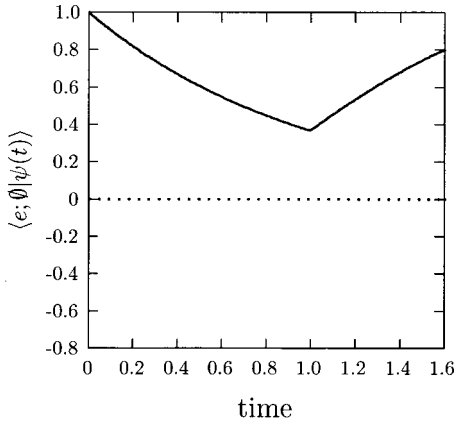

Fig. 5. As in Fig. 1, the amplitude $\langle e;\varnothing|\psi(t)\rangle$ for an initially excited atom to remain in the excited state, with no photons in the field. As in Fig. 1 the atomic resonance frequency matches that of an even-numbered mode of the cavity, but in this case the atom–field coupling is reduced to $\Omega=1$, and the radiation has returned to the atom before the spontaneous decay is complete. In this case, the decay is interrupted.

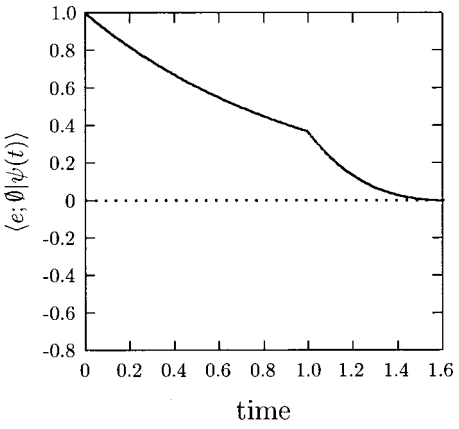

Fig. 6. As in Fig. 1, the amplitude $\langle e;\varnothing|\psi(t)\rangle$ for an initially excited atom to remain in the excited state, with no photons in the field, but in this case the atomic resonance frequency matches that of an odd-numbered mode of the cavity. The atom–field coupling is reduced to $\Omega=1$, and the radiation has returned to the atom before the spontaneous decay is complete. In this case, the decay is accelerated.

Fig. 3, which shows the intensity expectation value at $t=0.6$. At locations in the cavity such that $0.1<x<0.9$ the tail of the exponential intensity distribution is visible, but in the region of overlap, that is, $x<0.1$ and $x>0.9$, we see an interference pattern that is identical to the transient standing wave that would be formed by outgoing and reflected classical waves. The wavelength of this ''standing wave'' is exactly that which would be predicted for two counterpropagating classical waves at the resonance frequency of the atom. (In Fig. 3 the details of the standing wave in the region of overlap are on too fine a scale to be observed in a graph covering the whole cavity. Therefore Fig. 3 contains a graph covering the whole cavity with the standing-wave region indicated with shading, and an additional graph covering a much smaller region of the cavity.) The modulation depth of the standing wave is largest near the wall, where the counterpropagating waves have almost equal amplitudes, and decreases toward the middle of the cavity, where the right- and left-traveling components have different amplitudes due to the exponential envelope.

Our results for *outgoing* spontaneously emitted photons clearly exhibit the idea expressed by Mandel and Wolf[15] that for small photon numbers ''the very idea of an oscillatory field of definite phase … is meaningless according to the quantum theory of radiation.'' However, there is a sense in which spontaneously emitted single photons do have a definite internal phase that is revealed in regions where the photon can be said to interfere with itself due to reflection.

As discussed earlier, the excitation of a single mode would result in an expectation value of intensity that extends throughout the cavity with the sine-squared structure of the classical mode. The localized expectation value of intensity exhibited by the spontaneously emitted photon requires that the emitted field be comprised of many modes. The amplitudes for finding photons in the various modes are given by the coefficients $c_{g,j}(t)$ of Eq. (21). At times long compared to the atomic decay time, but before the atom is re-excited at $t=1$, our model gives the exact Lorentzian frequency spectrum that would be expected given the decay rate given by Eq. (29).[13,14]

Just as the photon exhibits a phase when interfering with itself, it has a phase relative to the emitting atom. In Figs. 1–3 the atom was assumed to be resonant with an even-numbered mode. If the atom is detuned from the even-numbered mode by $\delta=\Delta_c$ so that it is resonant with an odd-numbered mode, the amplitude for the atom to be in the excited state due to the reflection from the end walls changes sign, as illustrated in Fig. 4. This change can be understood in terms of classical phase arguments: if the atom is resonant with an even-numbered mode, it is located at a node of the resonant mode, and thus an integer number of half-wavelengths from the cavity walls. The reflected radiation must travel a total distance equal to an integer number of wavelengths before returning to the atom. If the atom is in resonance with an odd-numbered mode, it is located at an antinode, and the radiation must travel an integer number of wavelengths *plus* another half-wavelength before returning to the atom. Classically, the reflected radiation returns with a phase difference of π in the two cases, and classically the power delivered to a driven oscillator depends on the phase of the driving force relative to the phase of the oscillation. In the quantum case illustrated here, we see that the returning radiation can cause the *amplitude* for the atom to be in the excited state to either increase or decrease, depending on the path length traveled by the reflected radiation.

More generally, the revival in the excited state amplitude may have any phase. It is possible to use the Laplace transform techniques of Stey and Gibberd[14] to find analytical expressions for the amplitude for the atom to be in the excited state. One of us (ML) has extended the work of Stey and Gibberd to general values of the detuning parameter δ, and for times long enough that the effect of the first reflection is included, that is, $0\leqslant t\leqslant 2L/c$, the analytical form for the amplitude to find the atom in the excited state is[16]

$$c_e(t)=\langle e;\varnothing|\psi(t)\rangle = e^{-\gamma t}+\Theta(t-L/c)e^{i\pi\delta/\Delta_c}e^{-\gamma(t-L/c)}(t-L/c), \qquad (30)$$

where as before γ is given by Eq. (29), and Θ is the step function that ''turns on'' at the time the radiation returns to the atom. (Subsequent reflections add additional terms that ''turn on'' at later times.)

The effect of the phase of the returning radiation on the probability to find the atom in the excited state is negligible in the cases illustrated in Figs. 4 and 1; the phase disappears upon the squaring of the amplitude. The effect is *not* negligible if the radiation returns to the atom before the initial excitation has died away. If the phase factor in the second term in Eq. (30) gives $+1$, then the decay can be interrupted before its completion, and if the factor gives -1, the atom can be driven to the ground state faster than it would be in a simple decay. Results for such cases are presented in Figs. 5 and 6.

Re-excitation of atoms by properly phased reflections is at the origin of many quantum optical phenomena. In cavities that are short enough that the reflections from the end walls return before the atom has decayed significantly, that is, $L/c\ll 1/\gamma$, returning reflections interrupt the decay before it is even identifiable on the scale of Figs. 5 and 6, and subsequent returns continue to modify the decay. For returning reflections phased as in Fig. 6, the interrupted decay makes a transition to sinusoidal oscillation, known as vacuum Rabi oscillation. This transition has been investigated in Refs. 17 and 18. For returning reflections phased as in Fig. 5, the decay can be inhibited altogether. The total inhibition of de-

cay is an artifact of the model one-dimensional cavity, but inhibited decay in more realistic cavities has been studied theoretically and experimentally. Early work on the modification of the lifetimes of fluorescent dye molecules near reflecting surfaces is summarized in Ref. 19, and more recent studies of modified decays, mostly in the context of cavity quantum electrodynamics, are reviewed in Refs. 20–22.

V. CONCLUSION

In elementary treatments of modern physics, photons are treated as objects with a dual wave–particle nature. In this paper we have presented a fully quantized model of spontaneous emission in which the emitted radiation is treated as an excitation of a quantum field. By calculating physical observables of the atom-field system using the machinery of a simple quantum field theory, we are able to elucidate more clearly the way in which classical behavior of radiation fields manifests itself in the quantum regime.

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APPENDIX

The calculations discussed in this paper are all straightforward once the eigenvectors and eigenvalues of the total Hamiltonian of Eq. (14) have been determined. If we represent the basis vectors by the column vectors

$$|e;\varnothing\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad |g;-n\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \tag{31}$$

$$|g;-n+1\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \text{ etc.,}$$

and we label the constants giving the coupling of the atom to the mode with label j in the following way,

$$f_j \equiv \Omega \sin\frac{(j_0+j)\pi x_a}{L}, \tag{32}$$

then the matrix representation of the Hamiltonian is given by

$$\hat{H} \rightarrow \hbar \begin{pmatrix} 0 & \cdot & f_{-2} & f_{-1} & f_0 & f_1 & f_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{-2} & \cdot & -2\Delta_c+\delta & 0 & 0 & 0 & 0 & \cdot \\ f_{-1} & \cdot & 0 & -\Delta_c+\delta & 0 & 0 & 0 & \cdot \\ f_0 & \cdot & 0 & 0 & \delta & 0 & 0 & \cdot \\ f_1 & \cdot & 0 & 0 & 0 & \Delta_c+\delta & 0 & \cdot \\ f_2 & \cdot & 0 & 0 & 0 & 0 & 2\Delta_c+\delta & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{33}$$

We have calculated the eigenvalues and eigenvectors of this matrix in two ways. The simplest method is to use a computer package like MATHEMATICA or MAPLE. Sample MATHEMATICA code is illustrated at the end of this Appendix, with representative plots from a 300×300 matrix representation of the Hamiltonian. (In the case of an atom at the center of the cavity, half of the modes do not couple to the atom, so the matrices in the sample program are actually unnecessarily large.) Matrices of this size give very good results for the complex amplitudes for the individual basis states. Although they produce results that give the qualitative behavior of the expectation value of the square of the field, accurate representations near the abrupt rises at the edges of the pulses of expectation value require more terms to be included.

It is possible to derive an analytical form of the eigenvalue equation in the limit of an infinite number of modes following the methods used by Ligare and Becker.[13] For example, in the case of an atom located at the center of a cavity ($x_a = 0.5$) the eigenvalue equation is

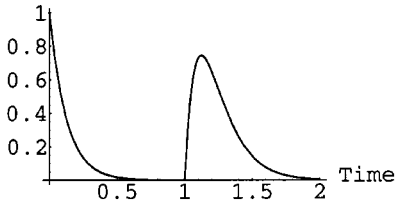
$$\frac{\pi\Omega^2}{2\Delta_c^2}\cot\frac{1}{2}\left(\frac{E}{\hbar\Delta_c}-\frac{\delta}{\Delta_c}+1\right)=\frac{E}{\hbar\Delta_c}. \tag{34}$$

The roots of this equation are bracketed by the regularly spaced singularities of the cotangent function, and it is easy to calculate thousands of roots in a few seconds of computer time using the simplest of root-finding algorithms. Explicit expressions for the elements of the eigenvectors in terms of the energy eigenvalues can be derived as in Ligare and Becker.[13] A program in a low-level language that uses such expressions is much faster than the MATHEMATICA program listed below, but the MATHEMATICA code is much more transparent.

Sample MATHEMATICA program

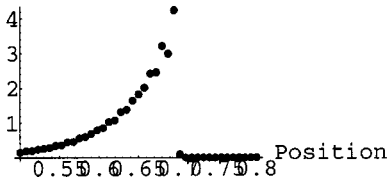
```
Clear[nmodes, dim, Ω, xa, Δ, j0, δ, h, vals, vecs, atomamp, photonamp, e2, x,
 t, j, l, xmin, xmax, np, dx]

nmodes = 299;     (* Number of radiation modes to include in calculation *)
dim = nmodes + 1; (* Dimension of Hamiltonian matrix                      *)
Ω = 4.;           (* Atom-field coupling constant                        *)
xa = .5;          (* Position of atom within cavity                      *)
Δ = π;            (* Angular frequency spacing between cavity modes      *)
j0 = 10000;       (* Even-numbered mode closest to resonance             *)
δ = 0;            (* Detuning of mode j0 from atomic resonance           *)

(* Fill Hamiltonian matrix.  See Eqs. 14 & 33 in text  *)
h = Table[0, {i, dim}, {j, dim}];
Do[{h[[1, j]] = h[[j, 1]] = Chop[ Ω Sin[(j0 + -dim / 2 - 1 + j) xa π]],
     h[[j, j]] = (-dim / 2. - 1 + j) * Δ + δ},
    {j, 2, dim}];

{vals, vecs} = Eigensystem[h]; (* Calculate eigenvalues and eigenvectors *)

(* Amplitude to find the atom in the excited state with no radiation    *)
(* modes excited.  See Eq. 22 in text.                                  *)
atomamp[t_] = ∑_{q=1}^{dim} Exp[-I vals[[q]] * t ] * vecs[[q, 1]] ^ 2;

 (* Calculate amplitude to find photon in mode detuned by angular frequency *)
 (* jΔ from central mode (and atom in ground state.  See Eq. 23 in text.    *)
photonamp[t_, j_] := ∑_{q=1}^{dim} Exp[-I vals[[q]] * t ] *
   Conjugate[vecs[[q, 1]] * vecs[[q, dim / 2 + 1 + j]]];

(* Calculate expectation value of the square of the electric field operator *)
(* See Eq. 25 in text.                                                      *)
e2[x_, t_] = ( Abs[ ∑_{l=-(nmodes-1)/2}^{(nmodes-1)/2} photonamp[t, l] Sin[(j0 + l) π x]] ) ^ 2;

Plot[Abs[atomamp[t]], {t, 0, 2}, PlotRange -> All,
 AxesLabel -> {"Time", "Atomic Excitation Amplitude"}];
```

Atomic Excitation Amplitude




```
xmin = .5;
xmax = .8;
time = .2;
np = 40;    (* Number of points to plot  *)
dx = (xmax - xmin) / np;
ListPlot[Table[{xmin + i * dx, e2[xmin + i * dx, time]}, {i, 0, np}],
            PlotStyle -> PointSize[.03],
            AxesLabel -> {"Position", "Expectation Value of E^2"}];
```

Expectation Value of E^2



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