Atomic decay and re-absorption in a one-dimensional cavity: Analytical results

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Abstract

I use Laplace transforms to find a complete analytical solution to the problem of a single two-level atom interacting with the quantized modes of a one-dimensional multimode optical cavity. The techniques were used by Stey and Gibberd [Physica, **60**, 1-26 (1972)] on several model Hamiltonians, and this paper is essentially a translation of one of their model Hamiltonians into the language of quantum optics. The results of this paper are a complement to the numerical work presented in Ligare and Becker [Am. J. Phys., **63**, 788-796 (1995)] and Ligare and Oliveri [Am. J. Phys., **70** 58-66 (2002)].

Simple fully quantized models of spontaneous emission and re-absorption have been investigated previously by the author [1,2]. In this paper I show how Laplace transforms can be used to find analytical solutions to the previously considered problems using the techniques introduced by Stey and Gibberd [3]. In this note I develop a Hamiltonian for a quantum optical system that is identical to a Hamiltonian considered in their paper. Additional mathematical details are presented in reference [3].

The system considered consists of a single two-level atom located at the position x_a in an optical cavity of length L. The atom is coupled to all of the quantized radiation modes of the cavity. The angular frequencies of the modes are given by

$$\omega_j = (j_0 + j)\pi c/L, \tag{1}$$

where j_0 is the label for the even-numbered mode closest to the atomic resonance, $\omega_{\rm eg}$. The modes are evenly spaced with angular frequency separation

$$\Delta_c = \frac{\pi c}{L} \tag{2}$$

and the detuning from atomic resonance of the mode with $j = 0$ is

$$\delta = \omega_0 - \omega_{\rm eg}. \tag{3}$$

I make the standard rotating-wave and electric-dipole approximations, so that the Hamiltonian is

$$\begin{aligned} H &= H_{\rm atom} + H_{\rm field} + H_{\rm interaction} \\ &= \hbar\omega_{\rm eg}\sigma_3 + \sum_j \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2}\right) + \sum_j \hbar\left(g_j\sigma_+ a_j + g_j^*\sigma_- a_j^\dagger\right) \\ &= \hbar\omega_{\rm eg}\sigma_3 + \sum_j \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2}\right) + \sum_j \hbar\left(\Omega\sigma_+ a_j + \Omega^*\sigma_- a_j^\dagger\right) \sin\left[(j_0 + j)\frac{\pi x_a}{L}\right], \end{aligned} \tag{4}$$

where σ_i are the elements of the atomic pseudospin operators, and the strength of the coupling of the atom to the field is characterized by

$$\Omega = d\left(\frac{\omega_{\rm eg}}{2\hbar\epsilon_0 V}\right)^{1/2}. \tag{5}$$

The sum over j includes all the modes of the cavity, and for $\omega_{\rm eg} \gg \Delta_c$ this sum over j will be effectively a sum of an infinite number of terms. For more discussion of the Hamiltonian see references [1,2].

In the remainder of this paper I will consider the specific case of an atom at the center of the cavity, i.e., $x_a = L/2$ (although it is possible to treat other positions $x_a = L/m$, where m is an integer, in a similar manner). In this case half of the terms in the interaction Hamiltonian go to zero, because the even numbered modes have nodes at the center of the cavity.

The basis states of the combined atom-field system are given by the direct product of the atom states and the field states:

- $|e;\emptyset\rangle$: Atom in the excited state, no photons in the radiation field,
- $|g;j\rangle$: Atom in the ground state, one photon in the radiation field with a detuning from the even-numbered mode closest to resonance given by $\omega_j - \omega_0 = \omega_j - (\omega_{\rm eg} + \delta) = j\Delta_c$. (The detuning of the photon from the zero-field atomic resonance frequency is $\omega_j - \omega_{\rm eg} = j\Delta_c + \delta$.)

In what follows the initial state of the system will be a state with an excited atom and no photons, that is,

$$|\psi(0)\rangle = |e;\emptyset\rangle. \tag{6}$$

Because the basis states with well defined photon numbers and atomic energies are *not* eigenstates of the total Hamiltonian, the state of the system evolves into a linear combination of the eigenstates, which can be written as

$$|\psi(t)\rangle = c(t)|e;\emptyset\rangle + \sum_j b_j(t)|g;j\rangle. \tag{7}$$

The Schrödinger equation gives the following set of coupled differential equations:

$$i\dot{c} = \sum_j g_j b_j \tag{8}$$

$$i\dot{b}_j = (j\Delta_c + \delta)b_j + g_j^* c \tag{9}$$

Taking the Laplace transform of these equations turns the coupled differential equations into coupled algebraic equations for the transform variables $\tilde{c}(s)$, $\tilde{b}_j(s)$:

$$i(s\tilde{c}(s) - 1) = \sum_j \tilde{b}_j(s) g_j \tag{10}$$

$$is\tilde{b}_j(s) = \tilde{c}(s) g_j^* + \tilde{b}_j(s)(j\Delta_c + \delta) \tag{11}$$

Solving this set of algebraic equations for $\tilde{c}$ gives

$$\tilde{c}(s) = \left[s + i \sum_j \frac{|g_j|^2}{(is - \delta) - j\Delta_c} \right]^{-1}. \tag{12}$$

For an atom at the center of the cavity the square of the coupling coefficient $|g_j|^2$ is zero for all even j, and Ω^2 for all odd j. Using this fact, the sum in Eq. 12 can be written

$$\tilde{c}(s) = \left[s + i \frac{|\Omega|^2}{2\Delta_c} \sum_{k=0,\pm1,\pm2,\ldots} \frac{1}{\left(\frac{is+\Delta_c-\delta}{2\Delta_c}\right) - k} \right]^{-1}. \tag{13}$$

In the case in which the atomic transition frequency is much greater than the fundamental frequency, i.e., $\omega_{eg} \gg \Delta_c$, the sum in Eq. 13 is approximately the sum of an infinite number of terms. Using a trigonometric identity [4] this equation can be written

$$\tilde{c}(s) = \left[s + i \frac{\pi|\Omega|^2}{2\Delta_c} \cot\left(\pi \frac{is + \Delta_c - \delta}{2\Delta_c} \right) \right]^{-1}. \tag{14}$$

For convenience I make the definitions

$$\gamma = \frac{\pi|\Omega|^2}{2\Delta_c} \tag{15}$$

and

$$r = \frac{\delta}{\Delta_c}. \tag{16}$$

I then rewrite Eq. 14 in terms of exponentials, and expand the result in powers of $\exp(-\pi s/\Delta_c)$, giving

$$\tilde{c}(s) = \frac{1}{(s+\gamma)} + \frac{2\gamma \exp\left[-i\pi(r + s/\Delta_c)\right]}{(s+\gamma)^2} - \frac{2\gamma(s-\gamma)\exp\left[-2\pi(ir + s/\Delta_c)\right]}{(s+\gamma)^3} + \cdots \tag{17}$$

Taking the inverse Laplace Transform of Eq. 17 term-by-term gives

$$c(t) = \exp(-\gamma t) + \Theta\left(t - \frac{L}{c}\right) 2\exp(-i\pi r) \exp\left[-\gamma\left(t - \frac{L}{c}\right)\right] \gamma\left(t - \frac{L}{c}\right)$$
$$-\Theta\left(t - \frac{2L}{c}\right) 2\exp(-i2\pi r) \exp\left[-\gamma\left(t - \frac{2L}{c}\right)\right] \left[\gamma\left(t - \frac{2L}{c}\right) + \gamma^2\left(t - \frac{2L}{c}\right)^2\right] + \cdots \quad (18)$$

where Θ is the unit step function. The successive terms in this expression all "turn on" at successive multiples of the round trip time of flight for photons from the atom to an end mirror and back to the atom.

REFERENCES

[1] M. Ligare and S. Becker, "Simple soluble models of quantum damping applied to cavity quantum electrodynamics," Am. J. Phys. **63**, 788-796 (1995).

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[4] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1965)