Atomic decay and re-absorption in inhomogeneous one-dimensional cavities: An example with an analytical solution

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Abstract

I use Laplace transforms to find the complete analytical solution for the quantum dynamics of a single two-level atom interacting with the quantized modes of an *inhomogeneous* one-dimensional multimode optical cavity. Lapleace transforms were used by Stey and Gibberd [Physica, **60**, 1-26 (1972)] on several model Hamiltonians, and this paper extends their techniques to cover decay in a cavity that has regions with two different indexes of refraction.

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Simple fully quantized models of spontaneous emission and re-absorption in homogeneous multimode cavities have been investigated numerically [1–3] and analytically [4]. In this paper I show how the Laplace transform techniques introduced by Stey and Gibberd [5] can be used to find analytical solutions to absorption and decay in a simple *inhomogeneous* cavity.

The system I consider consists of a single two-level atom located at the center of an optical cavity extending from $z = -L/2$ to $z = L/2$, as is illustrated in Fig. 1. The central portion of the cavity in which the atom is located is empty, and the symmetric end regions are filled with a linear dielectric material with a real index of refraction n. In this paper I will consider the specific case of a cavity in which the high-index regions extend from $-L/2$ to $-L/4$ and $L/4$ to $L/2$.

I quantize the standing-wave modes of the inhomogeneous cavity and then use a standard Hamiltonian of quantum optics to determine the time-evolution of the atom-cavity system. I make the standard rotating-wave and electric-dipole approximations, so that the Hamiltonian is given by [6]

$$\begin{aligned} H &= H_{\text{atom}} + H_{\text{field}} + H_{\text{interaction}} \\ &= \hbar\omega_{\text{eg}}\sigma_3 + \sum_j \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2} \right) + \sum_j \hbar \left(g_j \sigma_+ a_j + g_j^* \sigma_- a_j^\dagger \right), \end{aligned} \tag{1}$$

in which the sum extends over all modes of the cavity, and where σ_i are the elements of the atomic pseudospin operators, a_j and $a_k^\dagger$ are the lowering and raising operators for the j^{th} mode of the field with angular frequency ω_j, and the g_j gives the coupling between the atom and the j^{th} field mode. The frequencies ω_j are the frequencies of the classical standing-wave normal modes of the cavity. The spatial mode functions for the normal modes are sinusoidal,

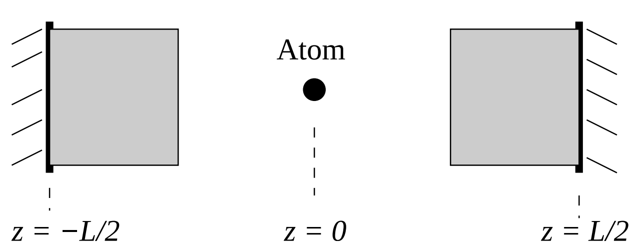

FIG. 1: Two-level atom in an inhomogeneous cavity. The central (unshaded) portion of the cavity is a vacuum, and the symmetric regions at the ends (shaded) have a real index n.

with wavelengths appropriate to the indexes of the regions. Using the boundary conditions of the cavity it is straightforward to show that the vacuum wave-vectors k for the even normal modes are given by the roots of the equation

$$\cos\left(\frac{kL}{4}\right)\cos\left(\frac{nkL}{4}\right) - \frac{1}{n}\sin\left(\frac{kL}{4}\right)\sin\left(\frac{nkL}{4}\right) = 0. \tag{2}$$

The coupling constants g_j depend on the amplitude of the modes at the center of the cavity. The odd modes all have nodes at the center of the cavity and therefore do not couple to an atom at $z = 0$, and need not be considered. The even modes will all have maxima (or minima) at the origin, and are normalized with constants $\mathcal{N}_j$ so that the expectation value of the energy corresponding to a single photon in mode j is $\hbar\omega_j$, giving

$$\mathcal{N}_j = \left[\frac{2}{2 + (n^2 - 1)\cos^2(k_j L/4)}\right]^{1/2}. \tag{3}$$

In the case in which the resonance frequency of the atoms $\omega_{\rm eg}$ is much greater than the frequency of the fundamental mode of the cavity, the explicit form of the coupling constants is

$$\begin{aligned} g_j &= \pm d\left(\frac{\omega_{\rm eg}}{2\hbar\epsilon_0 V}\right)^{1/2}\mathcal{N}_j \\ &= \pm\Omega\mathcal{N}_j, \end{aligned} \tag{4}$$

where in the last line I have defined

$$\Omega \equiv d\left(\frac{\omega_{\rm eg}}{2\hbar\epsilon_0 V}\right)^{1/2}, \tag{5}$$

where d is the dipole matrix element of the atomic transition, and V is the effective volume of the cavity.

I use as basis states the eigenstates of the atomic and free-field Hamiltonians, which I label

- $|e;0\rangle$: Atom in excited state, no photons in field,
- $|g;1_j\rangle$: Atom in ground state, one photon in field mode with frequency ω_j.

In what follows the initial state of the system will be a state with an excited atom and no photons, that is,

$$|\psi(0)\rangle = |e;0\rangle, \tag{6}$$

and the general state of the system is written as the linear combination

$$|\psi(t)\rangle = c(t)|e;0\rangle + \sum_j b_j(t)|g;1_j\rangle. \tag{7}$$

The Schrödinger equation gives the following set of coupled differential equations for the coefficients in Eq. (7):

$$i\dot{c} = \sum_j g_j b_j \tag{8}$$

$$i\dot{b}_j = \delta_j b_j + g_j^* c, \tag{9}$$

where δ_j is the difference between the cavity mode frequency and the atomic resonance frequency, i.e., $\delta_j = \omega_j - \omega_{\rm eg}$.

Taking the Laplace transform of these equations turns the coupled differential equations into coupled algebraic equations for the transform variables $\tilde{c}(s)$, $\tilde{b}_j(s)$:

$$i(s\tilde{c}(s) - 1) = \sum_j \tilde{b}_j(s) g_j \tag{10}$$

$$is\tilde{b}_j(s) = \delta_j \tilde{b}_j(s) + g_j^* \tilde{c}(s), \tag{11}$$

and solving this set of algebraic equations for $\tilde{c}$ gives

$$\begin{aligned} \tilde{c}(s) &= \left[s + i\sum_j \frac{|g_j|^2}{is - \delta_j} \right]^{-1} \\ &= \left[s + i|\Omega|^2 \sum_j \frac{|\mathcal{N}_j|^2}{is - \delta_j} \right]^{-1}. \end{aligned} \tag{12}$$

To demonstrate the manner in which the sums in Eq. (12) can simplify, I consider the specific case of $n = 2$, and take advantage of the periodicity that appears in Eqs. (2) and (3) for this value of the index. The left-hand side of Eq. (2) is periodic in the variable kL, and for any root $k'L$ there will be other roots at $k_m L = k'L + 4\pi n$, where n is an integer. There are three "families" of solutions to Eq. (2), all of them exhibiting this periodicity. One of the families has roots given by

$$k_m^{(0)} L = 2\pi m, \tag{13}$$

where m is an odd integer, and the other two families are located symmetrically with respect to the first family, with roots given by

$$k_m^{(\pm)} L = 2\pi m \pm \Delta k L, \tag{14}$$

where again m is an odd integer, and ΔkL is the smallest non-zero positive root of

$$\sin\left(\frac{\Delta kL}{4}\right)\cos\left(\frac{\Delta kL}{2}\right)+\frac{1}{2}\cos\left(\frac{\Delta kL}{4}\right)\sin\left(\frac{\Delta kL}{2}\right)=0. \tag{15}$$

The normalization factor will be constant within each of the "families" because of the periodicity of Eq. (3) in kL. The normalization constants for the "families" are

$$\mathcal{N}^{(0)} \;=\; 1 \tag{16}$$

$$\mathcal{N}^{(+)} \;=\; \mathcal{N}^{(-)}=\left(\frac{1}{2}\right)^{1/2}. \tag{17}$$

If we assume that the atomic resonance frequency exactly matches one of the cavity modes from the family with wave-vectors $k=2\pi m/L$, and we specialize to the case of $n=2$, then the Laplace transform $\tilde{c}(s)$ given by Eq. (12) becomes

$$\begin{aligned}\tilde{c}(s) \;&=\; \left\{s+i|\Omega|^2\left[\sum_j\frac{1}{is-j\frac{4\pi c}{L}}+\frac{1}{2}\sum_j\frac{1}{is-c\Delta k-j\frac{4\pi c}{L}}+\frac{1}{2}\sum_j\frac{1}{is+c\Delta k-j\frac{4\pi c}{L}}\right]\right\}^{-1}\\ &=\; \left\{s+i\frac{|\Omega|^2L}{4\pi c}\left[\sum_j\frac{1}{\frac{L}{4\pi c}is-j}+\frac{1}{2}\sum_j\frac{1}{\frac{L}{4\pi c}(is-c\Delta k)-j}\right.\right.\\ &\qquad\left.\left.+\frac{1}{2}\sum_j\frac{1}{\frac{L}{4\pi c}(is+c\Delta k)-j}\right]\right\}^{-1}.\end{aligned} \tag{18}$$

In the limit in which the atomic transition frequency is much greater than the fundamental frequency of the cavity, the sums in Eq. (18) can be approximated by extending them to include values of j from $-\infty$ to $+\infty$. This allows them to be written in terms of trigonometric functions [7] as follows:

$$\tilde{c}(s)=\left(s+i\frac{|\Omega|^2L}{4c}\left\{\cot\left[\frac{\pi L}{4\pi c}is\right]+\cot\left[\frac{\pi L}{4\pi c}(is-c\Delta k)\right]+\cot\left[\frac{\pi L}{4\pi c}(is+c\Delta k)\right]\right\}\right)^{-1}. \tag{19}$$

I define the quantity

$$\gamma=\frac{|\Omega|^2L}{c}, \tag{20}$$

and rewrite Eq. (19) in terms of exponentials. I then expand the result in powers of $\exp[-sL/(4c)]$, giving

$$\tilde{c}(s)=\frac{1}{(s+\gamma)}-\exp[-sL/(2c)]\frac{\cos^2\left(\Delta kL/4\right)}{(s+\gamma)^2}+\cdots \tag{21}$$

Taking the inverse Laplace Transform of Eq. (21) term-by-term gives

$$c(t)=\exp\left(-\gamma t\right)-\Theta\left(t-\frac{L}{2c}\right)2\exp\left[-\gamma\left(t-\frac{L}{2c}\right)\right]\gamma\left(t-\frac{L}{2c}\right)\cos^2\left(\Delta kL/4\right)+\cdots \tag{22}$$

where Θ is the unit step function. The first term represents decay at the vacuum rate γ [2, 4, 5]. The second term represents excitation due to the first reflection from the the high-index regions at the ends of the cavity, and "turns-on" at the time expected for a round-trip from the position of the atom to the interface with the high-index region and back. The additional terms that are not included in Eq. (22) turn on at successive intervals of $L/2c$.

In an earlier work [8] I translated the results of reference [5] into the language of quantum optics, and derived expressions for the excitation due to reflections in an empty cavity. The second term in Eq. (22) is identical to the the analagous term for an empty cavity except that the "turn-on" time is earlier (as expected), and the magnitude of the term is reduced by the factor $\cos^2(\Delta kL/4)$. Equation (15) gives

$$\cos^2(\Delta kL/4) = \frac{1}{3}, \tag{23}$$

and a reduction by this factor is exactly what is expected from classical considerations. The electric field reflection coefficient for the interface with the high-index regions is

$$R = \frac{n-1}{n+1} \longrightarrow \frac{1}{3}. \tag{24}$$

A classical field reflected from an interface with the high-index region is one-third of that reflected from a perfect mirror, and this leads to re-excitation of the atom (as expressed in the quantum amplitude $c(t)$) that is exactly one-third of the excitation in an empty cavity.

I note that the phase of the second term in Eq. (22) has been set by the condition that the atomic resonance exactly matches the frequency of one of the cavity modes with a wave-vector $k = 2\pi m/L$, where m is an odd integer. This condition is equivalent to the statement that the atomic resonance frequency is such that the round-trip distance from the atom to the high-index interface and back is equal to an odd number of half-wavelengths of the resonant light. Altering this condition corresponds to altering the phase of classical light returning to the atom, and this manifests itself in the quantum case in phase of the term in the complex quantum amplitude $c(t)$ that corresponds to re-excitation by the reflected field.

The calculations outlined in this paper are specific to the details of the cavity we considered, and must be modified to fit other physical situations. It is in principle possible to consider more complex dielectric structures, and it is also possible to consider cavities with additional atoms. In practice it is much easier to use numerical techniques like those discussed in references [2, 3, 9] when more complicated atom-cavity systems are considered.

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