# MATH 161 — Precalculus[1]
# Community College of Philadelphia


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[1]Materials produced with the support of the National Science Foundation through a grant to the Middle Atlantic Consortium for Mathematics and its Applications throughout the Curriculum (MACMATC).

# Math 161 — Chapter 1 Graphing and Lines Information

## 1.1 Introduction

In this chapter we review the basics of graphing and illustrate the use of graphs with two particular applications: stocks and motion. We then discuss lines, graphically and algebraically.

They say a picture is worth a thousand words. In this spirit, graphs are often used to convey information, in mass media as well as in mathematical and scientific work. The type of information they convey well is how one quantity changes depending on another. For example, graphs can be used to show how population in a particular place changes from year to year, or how an athlete's performance changes. To make a mathematical graph comparing two things you need to describe both in terms of numbers. For example, you could graph the population of Philadelphia over the last 20 years, but not way the color of the sky affects the civic mood. We don't have numbers for the latter (though people find ways to attach numbers to more and more things all the time). In studying a subject, say Philadelphia over the years, even restricting ourselves to things that can be measured with numbers, we need to understand that measurements are usually approximate. (At precisely what moment do we think we knew the exact population of Philadelphia, and did we check with all the hospitals about how many infants had been born in the preceding second?)

## 1.2 Stocks

Here's some background for our first application.

Stocks represent ownership of a fraction of a company or business. Total ownership is divided among many shares, which people buy and sell through brokers licensed to do such buying and selling, within an organization known as a stock exchange.

As an example, suppose a customer wants to buy 200 shares of XYZ stock which is traded on the New York Stock Exchange. S/he may place an order with a brokerage firm to buy the stock at the market price. The order is immediately sent to the firm's member on the floor of the exchange, who goes to a "post" for stock XYZ. (All trading must be done in a loud voice within a few feet of this post.) The specialist at the post announces at what price (the "ask" price) s/he is willing sell. The floor member may try to get a lower price from other floor members trying to sell stock at the post. If this effort is unsuccessful, s/he buys from the specialist. As a result of the bargaining, the price of the stock can go up or down throughout the day. (You may have seen television news reports of chaotic trading on particularly active days.)

### Reading Stock Quotations

Stocks are bought and sold every business day on various stock exchanges. The two usually listed in the newspapers are the New York Stock Exchange (NYSE) and the NASDAQ (National Association of Security Dealers Automated Quotations). (NASDAQ dealings are done by computer.) There are many stocks, and to save space the information about each company's stock is listed in columns with abbreviated headings. For our purposes we need to know the name of the stock (under "Stock") and the closing price (under "Last", in the second column from the right). The column on the far right gives the change in the price of the stock from the preceding business day.

## 1.3 Velocity

For our second application you need to know the following.

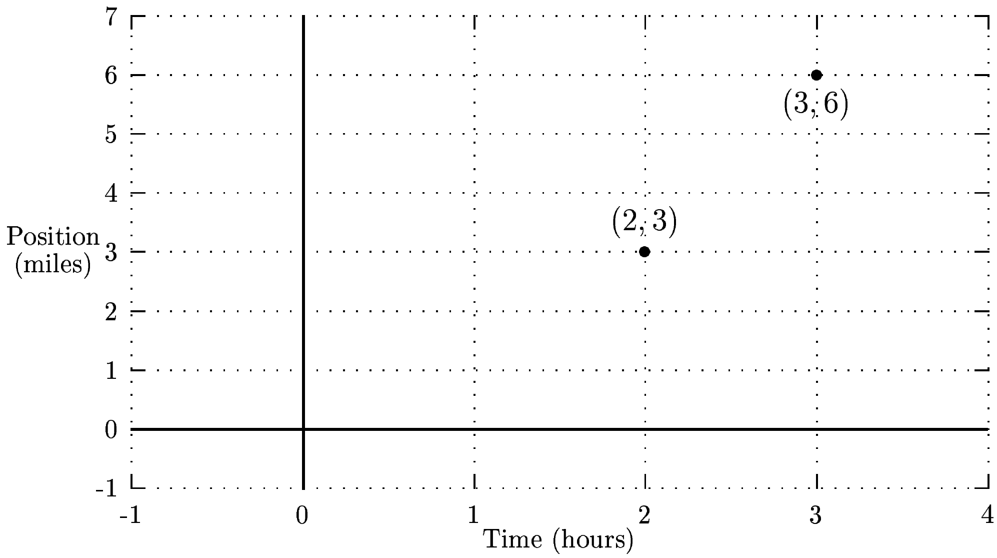
Speed is defined to be change of position over time. For example, if you walk 7 miles in 2 hours your speed is $7/2 = 3.5$ miles per hour. Velocity takes direction into account. By definition, Velocity $= \frac{\text{distance}}{\text{time}}$, and in discussing velocity we assume position is measured in a particular direction, from a particular starting point, and motion is in the chosen direction or its opposite (no wandering about the landscape). If you travel along a path with mile markers your velocity is taken to be positive if you travel from the lower numbers to the higher, and negative if you go the other way. Thus if you walked three miles starting at 1 p.m. and ending at 2 p.m., going from, say, Mile 7 to Mile 10, your velocity is 3 miles per hour (mph). If in the next hour you go back from Mile 10 to Mile 7, your velocity is -3 mph. Your change in position between 1 p.m. and 3 p.m. is zero, since you end up where you started, so your velocity is 0 mph. (=0 miles/2 hours). (Note this way of computing things does not measure the amount of exercise you got.)

## 1.4 Coordinate Systems and Plotting Points

Some types of information are usually given as pairs of numbers; for example, at 3 p.m. the temperature is 75 degrees (at Independence Mall, or wherever). It is common in mathematics and science to collect data in the form of ordered pairs of numbers, and often to plot these pairs as points in a coordinate plane, and then to look for patterns. The same sort of thing is done in sports and various other activities.

The graphs we use are produced by drawing two perpendicular lines, each called an *axis* (plural *axes*), one horizontal and one vertical. The point of intersection is called the *origin.* Units are marked at equal intervals along each axis. The size of the unit is called the *scale.* We do not always use the same scale on the vertical axis as one the horizontal axis, but it is essential to be consistent along each axis. If the distance between consecutive integers is not equal along an axis, serious distortions appear in the graphs we make. The horizontal axis is used to plot the *independent variable*, most commonly called $x$, and the vertical axis to plot the *dependent variable*, most commonly called $y$. As the names suggest, we generally think of the dependent variable as being determined by the independent variable. In giving the relationship between the two by a table we put the independent variable first, and in giving it by an algebraic formula we usually express the dependent variable in terms of a formula involving the independent variable(for example, $y = 2x + 1$). If we are describing a real-world situation we take the independent variable to be the one we regard as more fundamental than the other (time being a common example).

For example, if Juan the Walker is strolling along a highway with mile markers, he could tell you about his walk by listing pairs of numbers giving the *time* and his *position* at that time. He might be at mile marker 3 after 2 hours of walking, and at mile marker 6 after 3 hours of walking. These "time/position" pairs are denoted as $(2, 3)$ and $(3, 6)$, where the first number in the parentheses indicates the time and the second indicates his position at that time. Such pairs of numbers can be plotted as points in a plane. The first number, or *coordinate*, gives the horizontal position of the point (with respect to scale on the horizontal axis), and the second coordinate gives the vertical position (with respect to the scale on the vertical axis). axis. The two points $(2, 3)$ and $(3, 6)$ are plotted as follows:




The axes of graphs are usually labeled. If the horizontal axis is called the $x$-axis, then the first coordinate of a point is called its $x$-coordinate. If the vertical axis is called the $y$-axis, then the second coordinate of a point is called the $y$-coordinate. Axes are often labeled according to the interpretation of the points that are to be plotted. For example, instead of using $x$ and $y$, we might call the horizontal axis the *time* or $t$ axis, and the vertical axis the *position* or $p$ axis.

In the problems you do in this chapter you should notice patterns in the points you plot that tell you something about the real-world situation the graph is describing.

## 1.5 Lines

One of the things we do when graphing is to look for patterns: similarity among graphs with a certain type of algebraic formula, similarity among graphs that describe similar-seeming real-world phenomena, etc. In a particular graph, we look for a pattern to tell us how the graph would continue if we extended it further. The applications we consider in this chapter give us almost the opposite extremes of the possibilities. In the case of a stock, if we could look at its graph and predict what it would do tomorrow, we would be rich. Unfortunately, we can't. Graphs of the value of a stock versus time are almost always quite jagged, and always unpredictable.

On the other hand, motion at a steady rate, such as is considered in the exercises on walking and running, gives us a graph with a very clear pattern. It's a line. Lines are so predictable and comparatively easy to work with that in higher mathematics it is standard to think of other curves in terms of lines related to them, in particular their tangents. Thus it is important to understand lines throughly. The rest of this chapter is devoted to lines.

## Graphing a Line

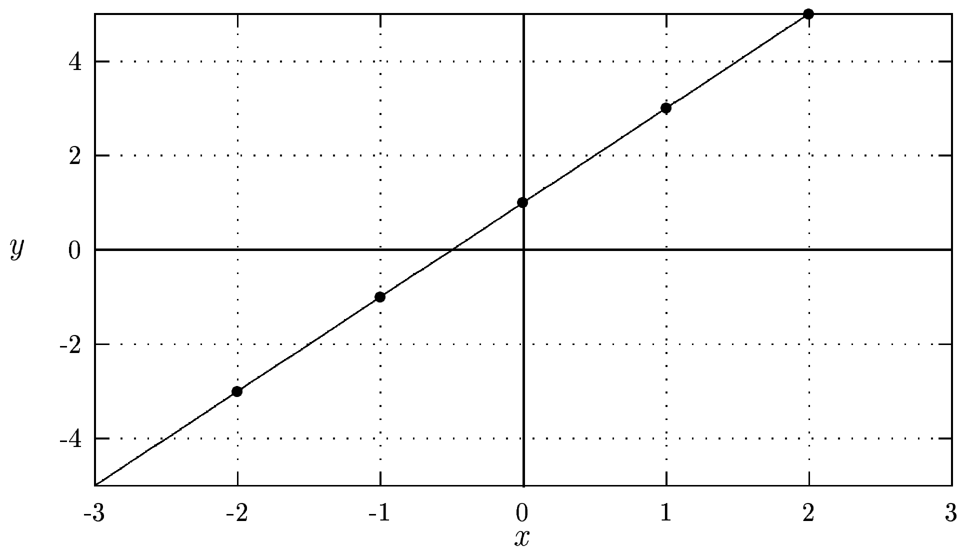
A line consists of a collection of points (lots of points!). As we shall see later, it is possible to predict from the type of an equation whether or not its graph is a line. The equation below has a line for a graph:

$$y = 2x + 1.$$

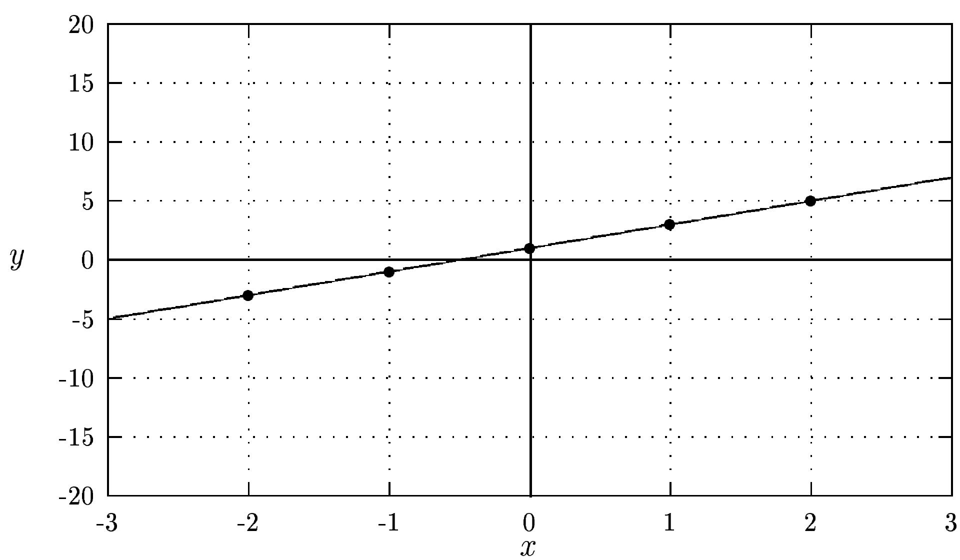
Each value assigned to the variable $x$ will give a value for $y$. For example, if $x = 3$, then $y = 7$. In general, the equation tells us to double the $x$ value and then add 1 to get the $y$ value. Often we pair the corresponding $x$ and $y$ values in a table such as the following:

| $x$ | $y$ |
|---|---|
| -2 | -3 |
| -1 | -1 |
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |
| 3 | 7 |
| 5 | 11 |
| 7.5 | 16 |
| 13 | 27 |

These pairs of $x$ and $y$ are certainly not the only pairs that satisfy the equation $y = 2x + 1$, but when representative points like these are plotted, they show the location of the line that represents all the points that satisfy the equation. The following graph shows some of the points and the single straight line that can be drawn connecting them:




If we change the scale on one or both axes the appearance of the line changes. But it still gives the same information. The same mathematical line is graphed below, but the vertical scale has been changed from the previous graph.




Though the graph of a line can change in appearance depending on the scales used, some properties remain constant and can be estimated easily from any graphical representation of the line.

The essential property of a line, its straightness, appears in every good graph. Whether the line rises or falls left-to-right is significant in applications because it tells whether the dependent variable increases or decreases as the independent variable gets larger. For example, it is of interest to know whether life expectancy increases, decreases, or stays the same with the amount of walking a person does, or the amount of a certain food s/he eats. Another question is how much difference it makes.

## Slope of a Line

The rate at which a line rises or falls can be determined from a graph, although it may look different in different graphs, depending on the scales used on the two axes. But for any line in a coordinate system, whatever the appearance of the graph, and whatever scale is used, no matter what two points on the line are taken, the ratio of the change in $y$ to the change in $x$ is always the same. The change in $x$ between two points is called the *run* and the change in $y$ the *rise*. This ratio is called the *slope* of the line.

**Definition:** The *slope* of a line is the unique number obtained from the following ratio, using any two points on the line:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}.$$

If the two points have coordinates $(x_1, y_1)$ and $(x_2, y_2)$ then the slope formula can be written

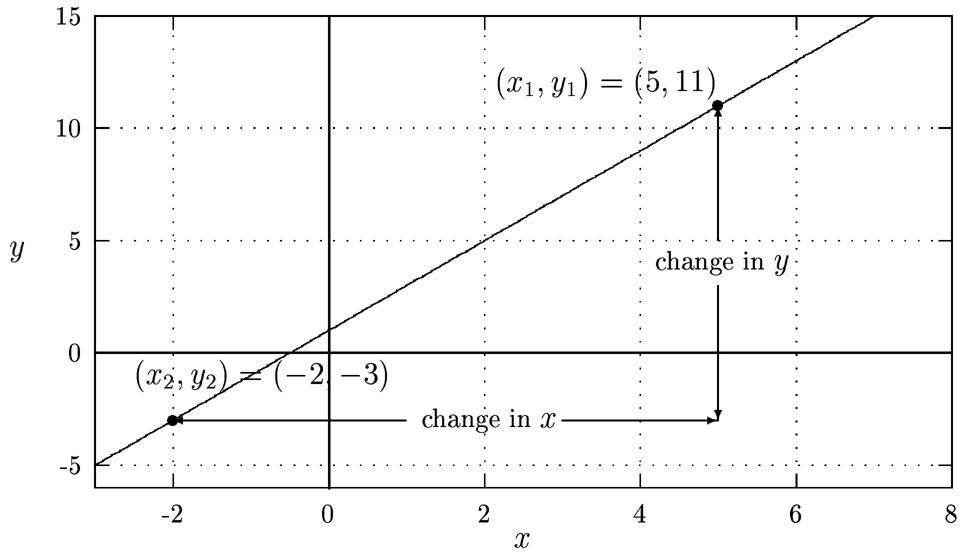
$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

**Example:** Consider the equation we have already graphed, $y = 2x + 1$. The points $(5, 11)$ and $(-2, -3)$ lie on the line, and if we regard $(5, 11)$ as $(x_1, y_1)$ and $(-2, -3)$ as $(x_2, y_2)$, and substitute these into the formula for the slope we see that $y$ changes by -14 and $x$ changes by -7 between these two points:

$$\begin{aligned} \text{slope} &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3 - 11}{-2 - 5} \end{aligned}$$

$$= \frac{-14}{-7}$$
$$= 2.$$

The graphical meanings of the quantities in this calculation are illustrated in the graph below.



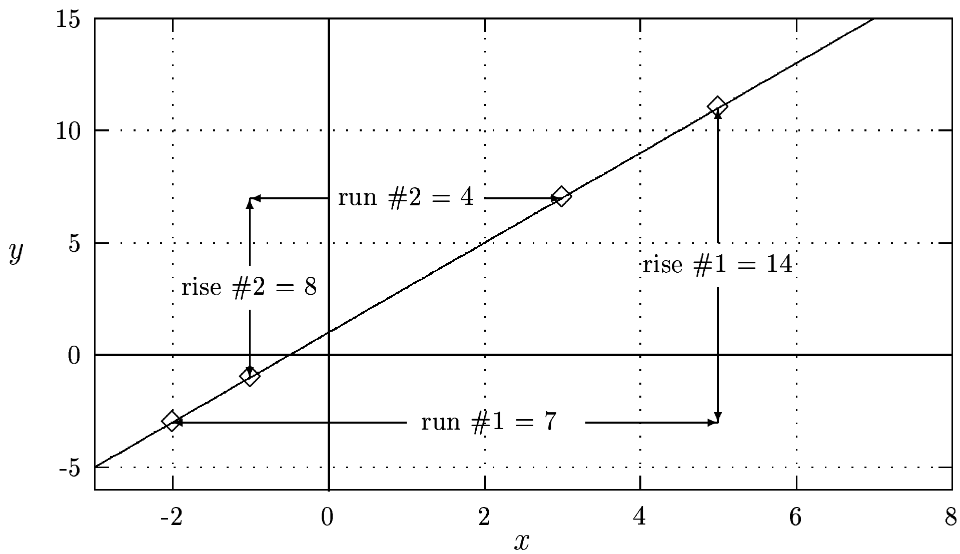

Sometimes the quantity $y_2 - y_1$ is given the symbol $\Delta y$, where $\Delta$ is the Greek letter *Delta*. Similarly $x_2 - x_1$ is given the symbol $\Delta x$.

Note that if we reversed the designation of the points, regarding $(5, 11)$ as $(x_1, y_1)$ and $(-2, -3)$ as $(x_2, y_2)$, we get the same result. This is true in every case (as long as the arithmetic is done correctly). (Many students find that the arithmetic is more likely to be right when there are fewer negative numbers involved, and often it's possible to see in advance that one way of designating the points is better than the other for this purpose.)

When we work with a line we get the same slope no matter which two points we pick. For example, the points $(-1, -1)$ and $(3, 7)$ also lie on the line $y = 2x + 1$, and if we calculate the slope we find

$$\text{slope} = \frac{7 - (-1)}{3 - (-1)}$$
$$= \frac{8}{4}$$
$$= 2.$$

which is the same result we got before. The slope is the same everywhere.




In geometric terms, we can say that if we take any two points on a line and draw a triangle formed by the segment between them and the segments for the rise and the run between them, this triangle is similar to any other such triangle formed using any two points on the line. The ratio of the rise to the run is the same in each case.

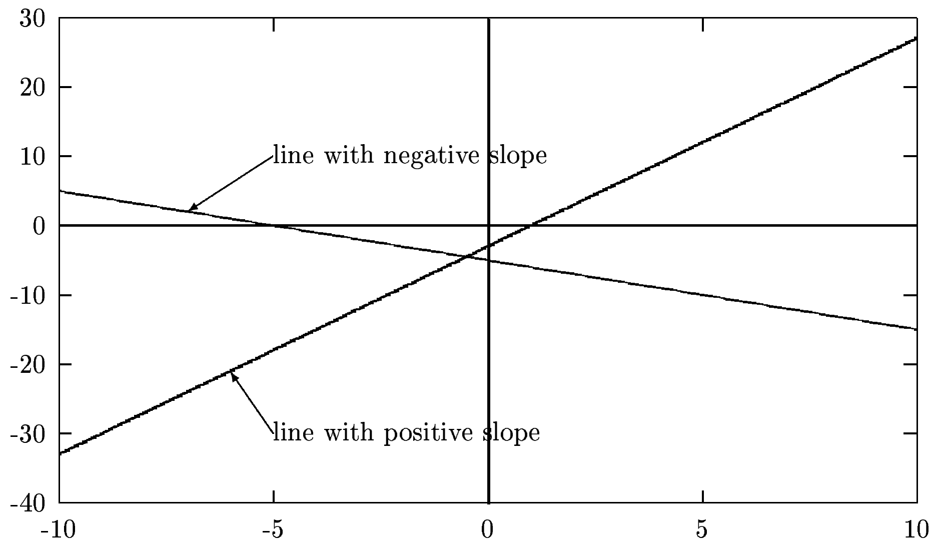
*Technical Note:* When evaluating the fractional expressions that arise in the formula for the slope of a line, you must be careful about the order of operations, especially when using a calculator or a computer. You must use parentheses around the entire numerator (or `Enter` it) and parentheses around the entire denominator in order for the calculator or computer to interpret your input correctly. For example, if you enter the data used in the example above as `-3-11/-2-5` your calculator will give you the answer $-2.5$, which is NOT correct. Your calculator assumed that you meant

$$-3 - \frac{11}{-2} - 5.$$

To get the correct answer you must enter the numbers as `(-3-11)/(-2-5)`.

The slope of a line is referred to as the *rate of change* of the variable $y$ with respect to $x$. (If you are looking at a linear relationship between two variables other than $y$ and $x$, then replace $y$ in the slope formula with the name of the variable plotted on the vertical axis, and replace $x$ with the name of the variable plotted on the horizontal axis.)

A positive slope tells you that as $x$ increases, $y$ increases. A negative slope tells you that as $x$ increases, $y$ decreases. The magnitude of the slope tells you how fast $y$ increases or decreases compared to $x$. For example, a slope of 3 indicates that $y$ increases three times as fast as $x$, or, put differently, for every unit $x$ increases, $y$ increases three units. A slope of $-2$ tells you that for every unit $x$ increases, $y$ decreases by 2. Lines with positive slopes are called *increasing* and those with negative slopes called *decreasing*.




## Equation of a Line

We now describe how to find an equation for any line, given two points on the line. By an equation of a line we mean an equation in $x$ and $y$ which is satisfied by points $(x, y)$ that lie on the line and by no other points in the plane.

The method of getting the equation uses the central fact about a line in a coordinate system: the slope is the same everywhere, and you can use any two points on the line to find it.

For example, suppose we want the equation of the line through (-1, 3) and (5, 12). First we find the slope:

$$\begin{aligned} \text{slope} = m &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12 - 3}{5 + 1} \end{aligned}$$

$$= \frac{9}{6}$$
$$= \frac{3}{2}$$

Thus $m = 3/2$.

Now suppose $(x, y)$ is another point on the line, and apply the slope formula to (-1, 3) and (x, y). This gives

$$\frac{3}{2} = \frac{y-3}{x+1}$$

Whatever point on the line is substituted for $(x, y)$, the right side of this equation must equal 3/2 because the slope is the same between any two points on the line. Thus we have an equation satisfied by every point of the line.

But not quite. The point $(5, 12)$ is on the line, but for this point the left side of the equation yields $\frac{0}{0}$, which is undefined (and in particular does not equal 1.5). We can get out of this easily enough by multiplying both sides of the equation by $x - 5$ to clear it of fractions:

$$\begin{aligned} (x-5)\frac{y-12}{x-5} &= 1.5(x-5) \\ y - 12 &= 1.5x - 7.5 \\ y &= 1.5x + 4.5 \end{aligned}$$

The method used in the example above can be applied to any two points, and the result can always be put in a similar form (as long as the line isn't vertical). We write the equation as

$$y = mx + b.$$

What about the constant $b$? If we substitute 0 into the equation, the $x$-term drops out and we get $y = b$. So the point $(0, b)$ is on the line. Any point with first coordinate 0 lies on the $y$-axis; i.e., $b$ is the *$y$-intercept*, so the second coordinate of the point of intersection of the line with the $y$-axis. (Note that a point whose second coordinate is zero lies on the $x$-axis, and a point of a line, or any other curve, that lies on the $x$-axis is the *$x$-intercept* of the line or curve.)

In summary, the equation of a line can be written in the form

$$y = mx + b,$$

where $m$ is the slope and $b$ is the $y$-intercept. Vertical lines are an exception and are discussed separately below. An equation for any non-vertical line can be put in the standard form given above, but it may require a little algebra to do so. For example, the equation $6x + 2y = 8$ can be converted to the form $y = -3x + 4$, and from this form you can see that this is the equation of a line with slope -3 and $y$-intercept 4.

Knowing the form of the equation of a line enables us to get the actual equation more quickly when we know either two points or alternatively one point and the slope. For example, starting with the two points above, $(-1, 3)$ and $(5, 12)$, we would find the slope to be $\frac{3}{2}$ (as in the original method). Then we use the fact that the equation must have the form $y = mx+b$, with $m = \frac{3}{2}$; that is, it looks like

$$y = \frac{3}{2}x + b.$$

To find $b$ we use the fact that for the right $b$ all points on the line, including the ones we know ($(-1, 3)$ and $(5, 12)$) satisfy this equation. If we substitute either point into the equation, we have an equation in which the only unknown is $b$. We solve for $b$ and we're done.

For example, if we substitute $(-1, 3)$ for $(x, y)$, we get $3 = 1.5(-1) + b$. Solving for $b$ we find $b = 4.5$.

The equation of the line is

$$y = 1.5x + 4.5$$

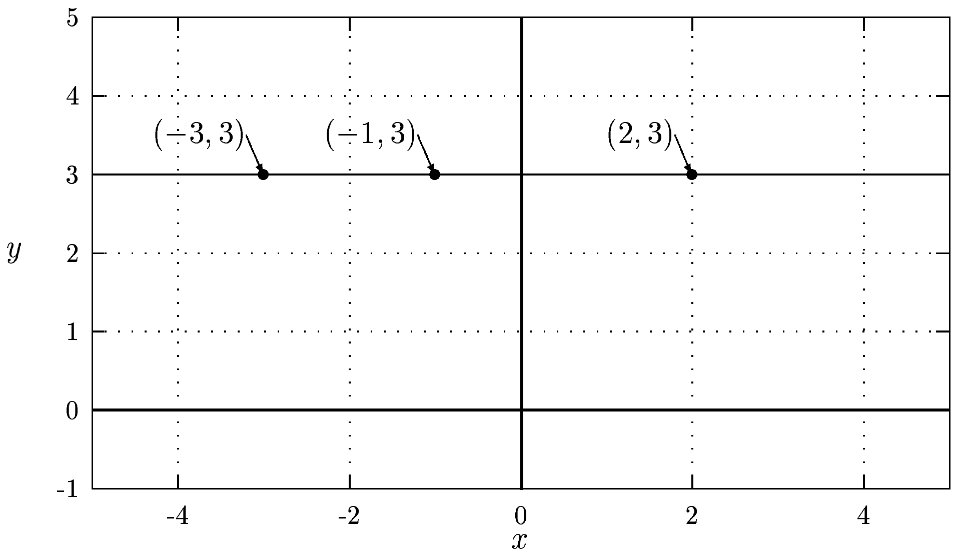
The other given point, $(5, 12)$ can be substituted as a check.

Note that this procedure for finding the standard equation of the line, which is quicker than the original one, depends on knowing what type of equation we can expect, and how the slope fits into this equation. In the study of mathematics, as in other endeavors, it pays to notice patterns.

## Some Special Cases and Relations

- Two lines in the plane are *parallel* if they never meet, no matter how far they are extended. Two lines in a coordinate system are parallel if and only if they have the same slope, or are both vertical.

- A *horizontal* line is one that is parallel to the $x$-axis. The second coordinate is the same for every point on such a line; so the rise is zero between any two points and hence the slope of the line is zero (since its numerator is 0). The equation of a horizontal line is of the form $y = b$, where $b$ is the second coordinate for all points on the line. (In the form $y = mx + b$, we have $y = 0x + b$, which simplifies to $y = b$.)




The graph of the line $y = 3$ is shown above. Note that applying the slope formula using, for example, the two points $(1, 3)$ and $(5, 3)$ we get $\frac{(3-3)}{(5-1)} = \frac{0}{4} = 0$. No matter what two distinct points of the line are used to compute the slope, they are of the form $(x_1, 3)$ and $(x_2, 3)$, and the slope is $\frac{(3-3)}{(x_2-x_1)} = 0$.

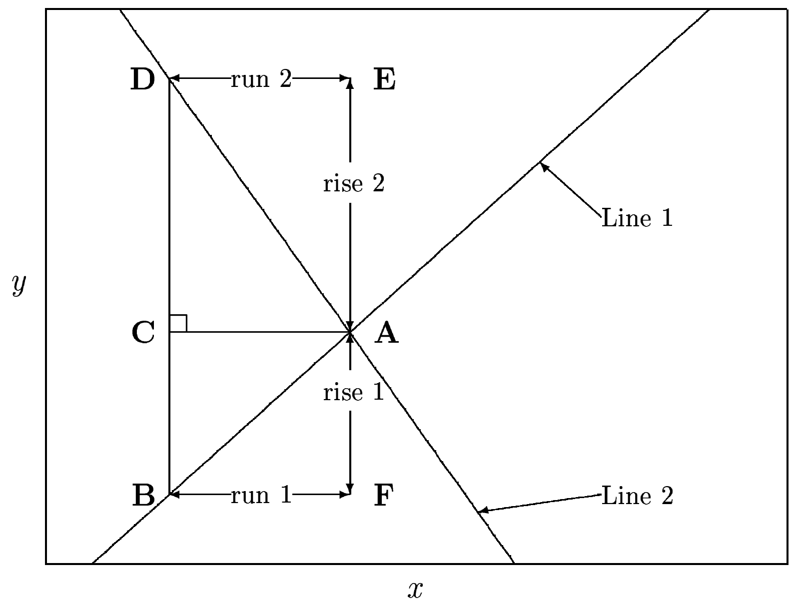
If we compute the equation the long way using $m = 0$ and any point $(x_1, 3)$ we have $\frac{(y-3)}{(x-x_1)} = 0$, and when we cross-multiply we get $(y-3) = 0(x - x_1)$, which simplifies to $y - 3 = 0$, or $y = 3$. Thus a horizontal line has an equation of the same form as a slanted line, except that it simplifies further than most. In a larger sense, the way to think about the equation of a horizontal line is in view of what an equation of a line is — it is an algebraic statement that describes the line. The description of a horizontal line in a coordinate system is that the second coordinate of every point on it is the same, regardless of what the first coordinate is.

- A *vertical* line is one that is parallel to the $y$-axis. Describing the line algebraically in simplest terms as we just did with a horizontal line, we note the first coordinate is always the same. Call this first coordinate $a$.

The line is described by the equation $x = a$ and $a$ is the first coordinate of each point on it. However, a vertical line is a special case in a way that a horizontal line is not. It does not have a slope. This is because in the case of a vertical line any attempt to get a slope using two points on the line yields a fraction with a denominator of zero. Division by zero being impossible, we don't get an answer. We do have an equation for the line, as just stated. However, this equation is *not* of the form $y = mx + b$. Vertical lines may be thought of as infinitely steep, like a cliff rather than a slope.

- Two lines are *perpendicular* if they intersect at right angles ($90°$ angles). Two lines in a coordinate system are perpendicular if and only if their slopes are negative reciprocals. That is, if one has slope $m$, the other has slope $-1/m$. For example, if one has slope 5 the other has slope -1/5. If one has slope 3/4, the other has slope -4/3.

  (The geometrically inclined can use the diagram below to see why this is true. Remember that in similar triangles, corresponding sides are proportional, and remember that in a right triangle the acute angles are complementary (that is, their sum is $90°$). Also, note that when taking the ratio of rise to run in order to get the slope we must consider direction: left-to-right gives a positive result as does lower-to-higher. The reverse direction of either gives a negative result. If you wish to try the proof, note that triangles $ACD$ and $DEA$ are congruent, as are triangles $ACB$ and $BFA$. Also, they are all right triangles. In addition, all these triangles are similar; hence their corresponding sides are proportional. Writing the proportions obtained from the similarity and involving the horizontal and vertical sides of the triangles leads to the result.


D
run 2
E
rise 2
Line 1
y
C
A
rise 1
B
run 1
F
Line 2
x