MATH 161 — Precalculus[1]
Community College of Philadelphia

Joanne Darken
Martin Ligare

[1]Materials produced with the support of the National Science Foundation through a grant to the Middle Atlantic Consortium for Mathematics and its Applications throughout the Curriculum (MACMATC).

Math 161 — Chapter 2
Functions
Information

2.1 Definition of a Function and Examples

The concept of function is central in mathematics. The definition is short, but it is the beginning of a way of thinking that has powerful and far-reaching results.

The working idea of a function is that it assigns to every item in one set exactly one item in another. The first set is called the *domain* of the function and the second the *range.* The elements of the domain are called inputs and the elements of the range outputs. The assignment of outputs to inputs can be made in various ways: by a verbal description, an algebraic formula, a table or a graph. In precalculus most functions are given by algebraic formulas, and we pay a good deal of attention to graphs. But the other ways of describing a function arise as well, and working with them helps you develop a good overall concept of how to think of a function.

In higher mathematics a function is defined as a set of ordered pairs in which no first element appears in two different ordered pairs. Think about this enough and you see that it is the same as saying each input has only one output. Most people find it hard to think of functions as ordered pairs, and we will not use this definition much. But because it may be helpful in understanding some ideas concerning graphs of functions, we include among examples some involving functions given as sets of ordered pairs.

Example 1: Some electric companies have a gizmo (called an itron) that checks meters remotely and assigns to each customer the appropriate bill. Each input (customer or account number) is assigned an output (amount owed) each month, and we can regard this assignment as a function with domain the set of customers (or their account numbers) and the range the set of amounts owed. (Of course, each billing produces a new function.) In the illustration below the domain is set of customers, shown by house number, and the range consists of the amounts of their electric bills. (Note that two customers are charged the same amount — this can happen.)

The electric company can keep records of this sort as tables (either in a computer or on paper) with a column for account numbers and a column for the current bill. Any such table can be regarded as a function, with the assignment to each account number an amount due. More generally, we may assign to any number another number or to any entry another entity.

Thus we may think of a function as a set of ordered pairs. In this way of describing a function, the domain is the set of first elements in the pairs, and the range is the set of second elements.

The billing in this example can be represented by the 2-row table below:

Address (City Street #)	101	103	105	107	109
Amount (in $)	37	23	41	41	29

The fact that no input is linked with more than one output means that no account is charged for two different amounts.

Example 2: We could take the domain of a function to be the set of students in a particular classroom (at a particular time, so there's no doubt about who's in the domain) and the rule to assign to each person the month of his or her birth. This is a function. It is a clear rule that assigns to each person exactly one month. The range is the set of months that occur as birthdays of those students (maybe all months, maybe not).

Example 3: Any well-defined set of people could be the domain for a variety of functions. For example, we could assign to each person enrolled at this institution his or her social security number. Each person has a social security number, and only one, so the rule gives a function. Or, we could assign to each person his/her current age.

Example 4: If we assign to each person enrolled at the institution the time he or she gets to school on Tuesdays, we do not have a function, because the rule doesn't give a clear-cut answer. Some people don't come in on Tuesdays, and most who do would come at somewhat different times on different Tuesdays. So we don't have an output for every input, and for some inputs (most or all that have outputs, probably), the output is not unique. Either problem means the rule doesn't give a function.

Example 5: Any table like the one in the previous example may give a function (as long as no input is linked with more than one output). In the following table

x	1	2	3	4	5	6
y	0	3	4	10	4	0

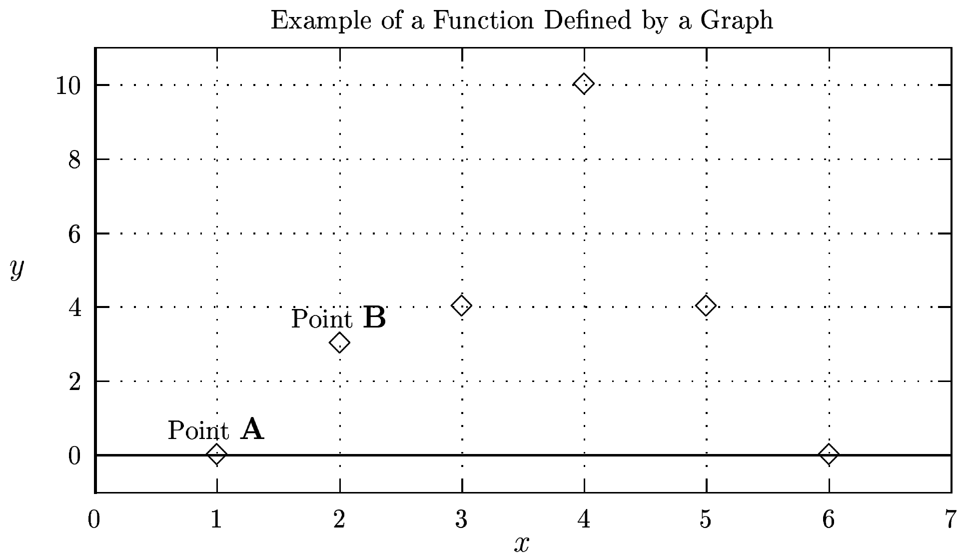
the top row is understood to give the input and the bottom row to give the output. The table could be written vertically, with inputs in the left column, and outputs in the right column. The function represented in the table could also be written as a set of ordered pairs. Using the usual curly brackets for sets, we rewrite the function defined in the table as
$\{(1,0),(2,3),(3,4),(4,10),(5,4),(6,0)\}$.

Example 6: The rule which assigns to every real number its square gives a function for any real number domain. We may choose for domain the set of all real numbers. If we take the domain to be all real numbers then the range is the set of all non-negative real numbers. (You never get a negative number for an answer when you square a real number.) If we take the domain to be the set $\{1,2,3\}$, the range is $\{1,4,9\}$.

Example 7: The rule $y = 2x + 3$ gives a function with domain all real numbers. For each x we choose, this rule assigns a value of y. For example, if $x = 4$, this rule assigns it a value $2 \cdot 4 + 3 = 11$. If we take all real numbers for the domain, then the range is also all real numbers. (If we choose to take a smaller domain, the range will change accordingly.)

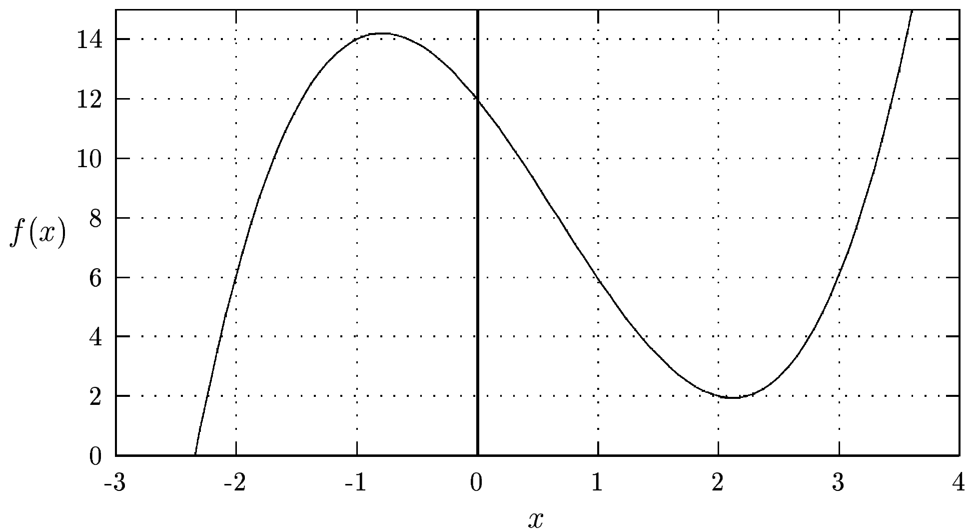
Example 8: If we use the rule that assigns to each number its square root, we can't apply it to all real numbers, because not every number has a (real) square root–only non-negative numbers do. So if we wanted to use the rule, we would have to use a domain that contains no negative numbers.

Example 9: A graph can give a function. The following graph defines a function for a domain of 6 values.



The coordinates of each point give an ordered pair. The function is considered to assign to each x-coordinate of the plotted points the value of the y-coordinate. For example, looking at point **A** we see that this function assigns to 1 (x-coordinate) the value 0 (y-coordinate), and looking at point **B** we see that it assigns to 2 the value 3. The function defined by this graph is actually the same function defined by the table in Example 52 . It could also be expressed as a set of ordered pairs: $\{(1,0),(2,3),(3,4),(4,10),(6,0),(5,4)\}$.

Example 10: A graph can also give a function defined at an infinite number of points. Suppose a function y is defined by the graph below; *i.e.*, it assigns to each number on the x-axis the y-coordinate of the point directly above it on the curve. For example, if x is 0, then y is 12, and if x is -2 then y is 6. The domain is the interval from $x = -2.37$ to $x = 3.61$ in which the graph is visible; other numbers aren't assigned y values.



There is a limit to the accuracy you can get with a graph. In constructing this graph, the authors used an algebraic formula, so can say with assurance what the answers are. If you just read the result from the graph without this advantage, you need to be aware that you may not get perfect accuracy. For example, you couldn't really be sure from this graph that when $x = -2$ that y is exactly equal to 6; maybe y equals 6.01, or some other number close to 6. But you can get good approximate answers this way. Sometimes information is given only graphically, and that's the best you can do.

In precalculus the functions we work with almost all have for their domain all the real numbers or some subset of the real numbers.

2.2 Functional Notation

It is often convenient to give functions names and to use the names in working with them. This is done with *functional notation.* For example, instead of writing

$$y = 2x + 3$$

we can write, using functional notation,

$$f(x) = 2x + 3.$$

This is read "f of x equals $2x + 3$."

Functional notation gives a convenient way of describing the output for a particular input. In this example, if the input is, say, 4, we write

$$f(4) = 2(4) + 3$$

and, doing the arithmetic, get

$$f(4) = 11.$$

This notation is standard, and easy to use, but there is a tendency when you first see it to think of multiplication, so be sure you understand that here, for example, f is not being multiplied by 4.

Functional notation also gives us a way of distinguishing among rules if we're working with more than one. Other letters commonly used to denote functions are g and h. We usually use the letter x to denote the input of a function, and y to denote the output. Other letters can be used; for example, t is commonly used instead of x if the input is time.

2.3 Ways of Presenting Functions

Tables

If the function f is given by the table

x	1	2	3	4	5	6
y	2	-3	8	1	0.75	π

we have that $f(1) = 2$, $f(2) = -3$, $f(3) = 8$, $f(4) = 1$, $f(5) = 0.75$, and $f(6) = \pi$.

The domain of this function is the set of numbers in the first row, and the range is the set of numbers in the second row. (Even in cases where you see a pattern, and could write an algebraic rule, be aware that if you do so you are creating a new function if you choose to expand the domain. The particular domain is part of the definition of the function.)

A table of this sort (an input, an output) appeared in Chapter 1 as a step in plotting a graph of the equation $y = 2x + 1$. In precalculus such tables most often appear as aids in graphing. (Usually they are produced by you on a sheet of your own paper rather than in the text, and are probably vertical rather than horizontal). They also arise in data collection processes, as when recording the closing price of a stock each day, or keeping track of a walker's position with the passage of time. Some tables in the exercises show variations that may mean that the information recorded does not actually give a function. If someone constructed a table with one of the oddities shown there it would probably come from absent-mindedness: a mistake in arithmetic or a misreading of information. But it might come from some mathematical or real-world issue that means we are not really going to get a function from this data. For example, if we make a table for the equation $x^2 + y^2 = 25$, the input $x = 3$ could have outputs $y = 4$ and $y = -4$. This equation does not describe a function.

Some points that satisfy $x^2 + y^2 = 25$						
x	3	3	-4	-4	0	0
y	4	-4	-3	3	-5	5

If we make a table using the temperature at Independence Hall as input and the time of day as output, we find that we get repetitions; e.g., it might be 72° at 1 p.m. and again at 4 p.m. This tells us that time of day is not a function of temperature — you can't tell time from a thermometer. We get a table that does not represent a function.

Time and temperature at Independence Hall						
temperature	72°	76°	76°	72°	67°	66°
time	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.	6 p.m.

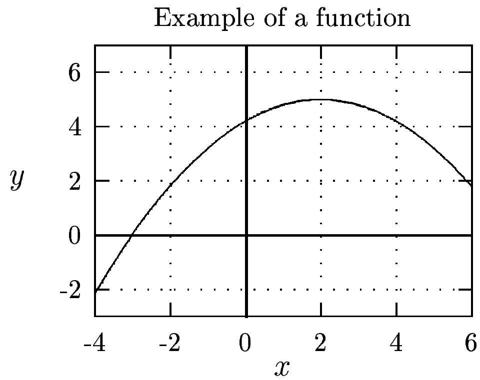
You can't tell temperature from a clock either, of course, but you can define the temperature as a function of time of day after picking a particular day — say July 4, 1776 — by using time as input and temperature as output.

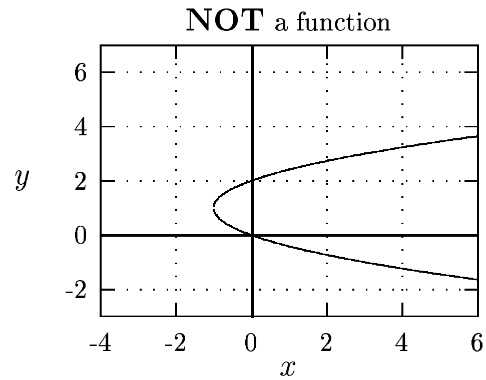
Graphs

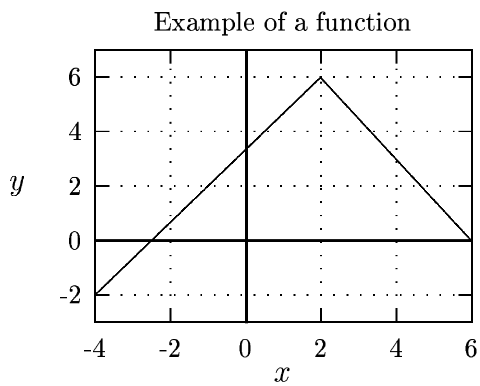
A graph can give information about a function in a form easy to take in. Its chief drawback is that due to the limitations of resolution of ink on paper or pixels on screen, the accuracy cannot be perfect, as it can with an algebraic formula or a table (depending on how the table arises). (With data gathered from the real world there are limitations on accuracy anyway, but it's better not to compound them. Don't throw away your tables after you've made your graph.)

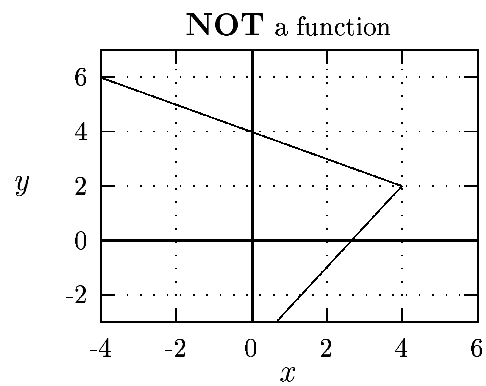
If a function is given by a graph, the horizontal axis is used for the input and the vertical for output. To read the output for a given input you go up or down from the input value till you get to the output point, then go horizontally to the y-axis to read the coordinate.

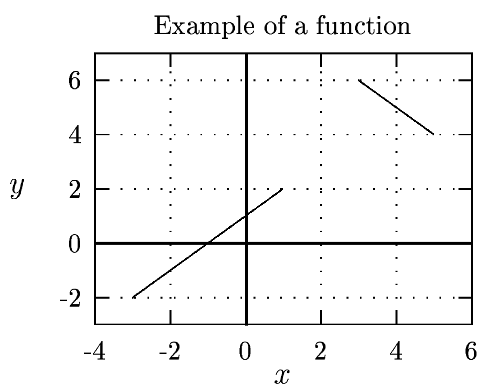
A graph may show a curve or other set of points that don't actually give a function. This happens if there are two or more points directly above or below the input value. This check is called the *vertical line test.*

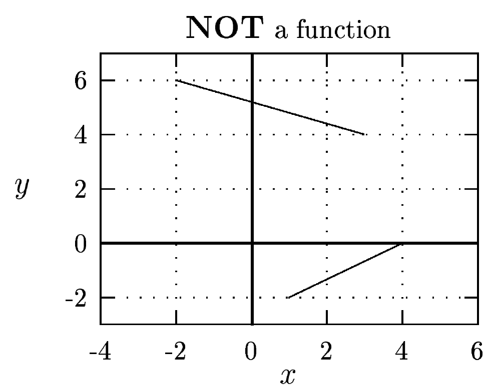












Among the pictures above are two in which the graph is not connected. Most but not all of the functions we work with in precalculus are connected; that is, their graphs can be drawn without lifting pencil from paper. Be aware that a graph need not be connected to represent a single function.

It may happen when a function is defined by a graph that some x values in the x interval shown do not have outputs. Assume that they are not in the domain.

A graph can give a much faster understanding of the overall behavior of a function than a chart or a formula. We often show graphs that are meant to convey the whole picture, even what's not shown. In this case the pattern needs to be clear so that the reader can extrapolate — not to get exact values but to see the pattern. We must also have a convention about whether the domain is understood to be what we see in the picture or some larger set (such as all real numbers). If a graph has an obvious pattern, it is customary to assume unless otherwise stated that it keeps on according to that pattern. For most of the functions we work with, this means the domain is assumed to be all real numbers.

Algebraic Formulas

Algebraic formulas carry us beyond the visible, and give exact rather than approximate outputs. You have already seen examples of functions given by algebraic formulas, and a description of how to evaluate such functions. With experience you will gain skill in determining important properties of a function from its formula.

If an algebraic formula gives the output in terms of an algebraic formula involving some of the usual algebraic operations ($+$, $-$, x, $\div$,$\sqrt{\ }$) on the input (x), delete then executing the steps of the formula leads to at most one answer, thus giving a function. (For the radical we need to recall the convention that a root without a sign in front of it is assumed to represent the positive root in cases where there are two roots. If we had a choice of outputs we wouldn't have a function.) A function may have a single output for all inputs; for example, $f(x) = 3$.

Sometimes a relation between two variables x and y is given by an equation not explicitly giving one variable in terms of the other; for example $2x + 3y = 6$, or $x^2 - y = 0$, or $x^2 + y^2 = 25$. If such an equation can be solved for y is terms of x without ambiguity then it gives y as a function of x, otherwise not. The first two equations in this paragraph can be so solved, giving $y = (6 - 2x)/3$ and $y = x^2$ respectively. However, in the last case we get $y = \pm\sqrt{25 - x^2}$, so there are inputs with more than one output (*e.g.*, the input $x = 3$ gives ± 4) and we do not have a function.

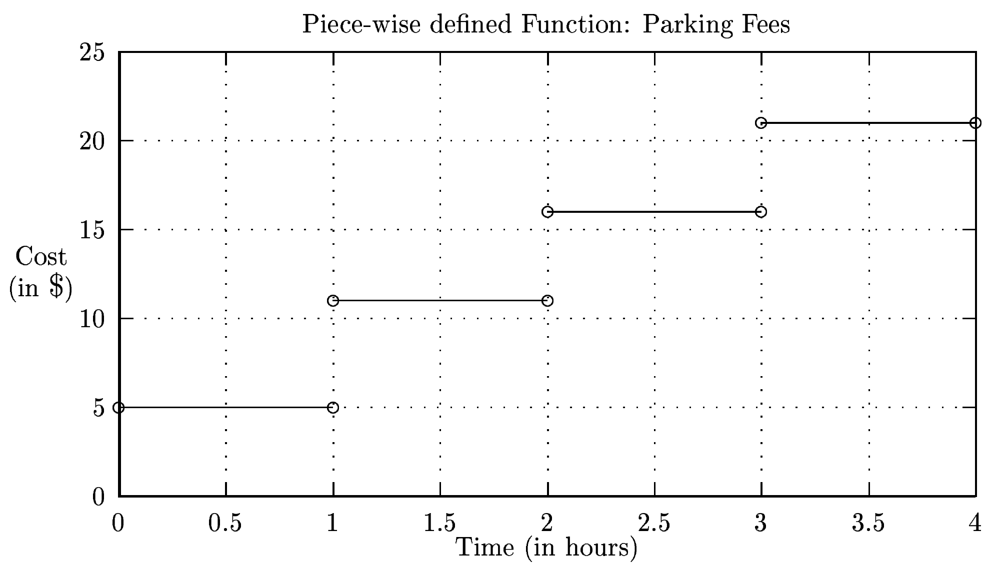
Sets of Ordered Pairs

A function consisting of just a few inputs with their outputs sometimes serves as a useful example, and we present some of these as sets of ordered pairs.

Such a function can also be presented by a table or graph. (Example 9 above contains the graph of the function consisting of the set of ordered pairs $\{(1,0),(2,3),(3,4),(4,10),(5,4),(6,0)\}$, and Example 5 gives the table for this function.) But our main interest is in functions given by algebraic formulas.

2.4 Piecewise-Defined Functions

A piecewise-defined function is one that is defined by different sub-rules on different sub-intervals of the domain. For example, parking lots and garages usually charge by the hour or part of an hour. The charge for the first hour or part of an hour might be $6.00, and for each additional hour or part of an hour an additional $5.00. A graph of this is a collection of horizontal pieces:



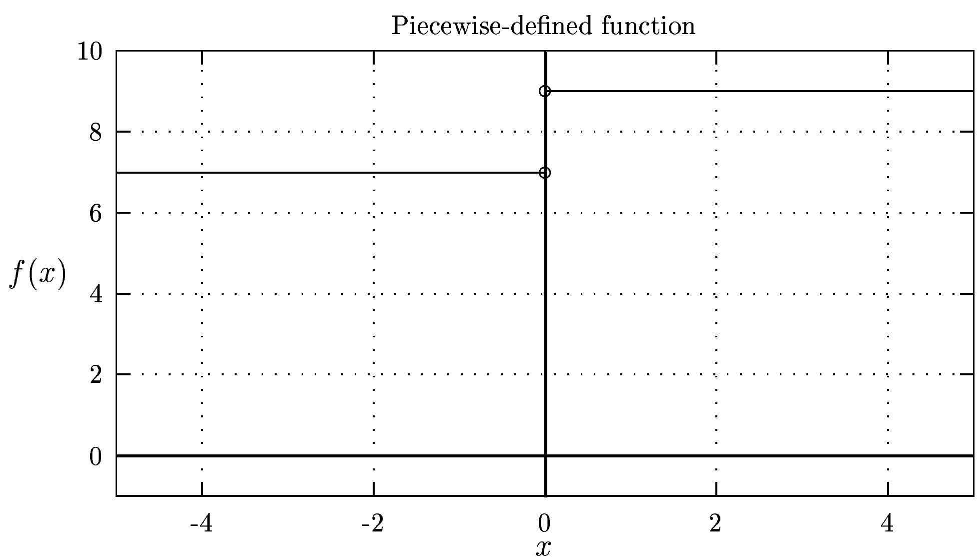
Most students find piecewise functions strange at first, perhaps because of a feeling that the rule shouldn't change as you go along. But this happens in many applications in mathematics and science as well as in everyday matters such as the parking lot example, and we must develop tools to deal with it.

We now introduce notation for a function given by different rules on different parts of the domain. For example, if we want our rule to be that if the input is less than 0 the output is 7, and if the input is greater than or equal to 0 the output is 9, we write

$$f(x) = \begin{cases} 7 & \text{for } x < 0 \\ 9 & \text{for } x \geq 0 \end{cases}$$

As usual with standard mathematical notation, this is very concise. Such brevity is a useful quality in the long run, but many people find this particular notation a little hard to get used to. It's saying what the sentence above says: if $x < 0$, then $f(x) = 7$, and if $x \geq 0$, then $f(x) = 9$. So to find the value of the function for a particular input, we look to see whether or not the input x is negative, and use the appropriate output. For example, $f(-5) = 7$, $f(-0.00017) = 7$, $f(-\pi) = 7$, $f(6) = 9$, $f(84,378.12) = 9$.

The graph of this function is shown below. Its domain is all real numbers (since no restrictions are given) and its range is the set $\{7, 9\}$.

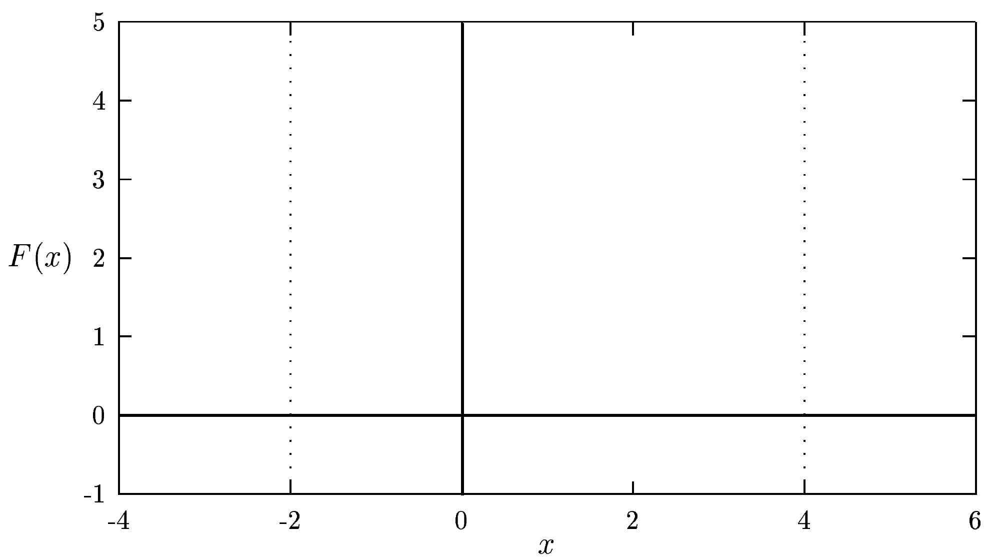


When graphing a piecewise function, it is useful to mark on the x-axis the values at which the rule changes. Then draw the graph for each rule. It may be helpful in some cases to sketch the graph for each rule over a larger interval than the one on which it applies, then erase the part you don't need. (This might be the case, for example, if you find it helpful to include the y-intercept in the sketch of one of the rules, even though it doesn't appear in the final result.)

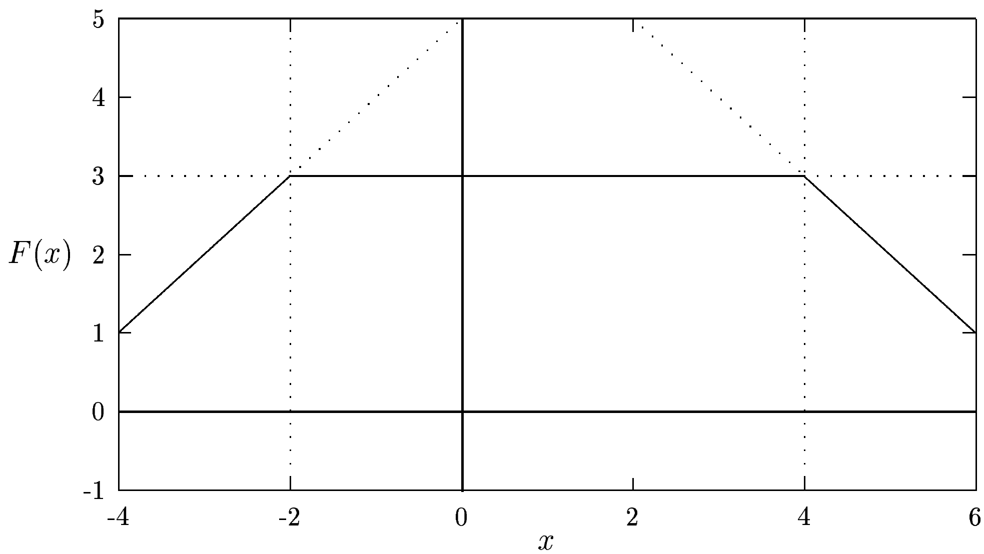
Example 11: Consider the piecewise function

$$F(x) = \begin{cases} x + 5 & \text{for } x < -2 \\ 3 & \text{for } -2 < x < 4 \\ 7 - x & \text{for } x > 4 \end{cases}$$

The rule to be used changes at -2 and 4, so draw a vertical line through these values on the x-axis.



Then graph each of the three rules: $y = x + 1$,$y = 3$, $y = 7 - x$. Since it is helpful in sketching a line to show the y-intercept, you may want to show a more complete sketch of each line than you need for the final graph — but draw lightly:

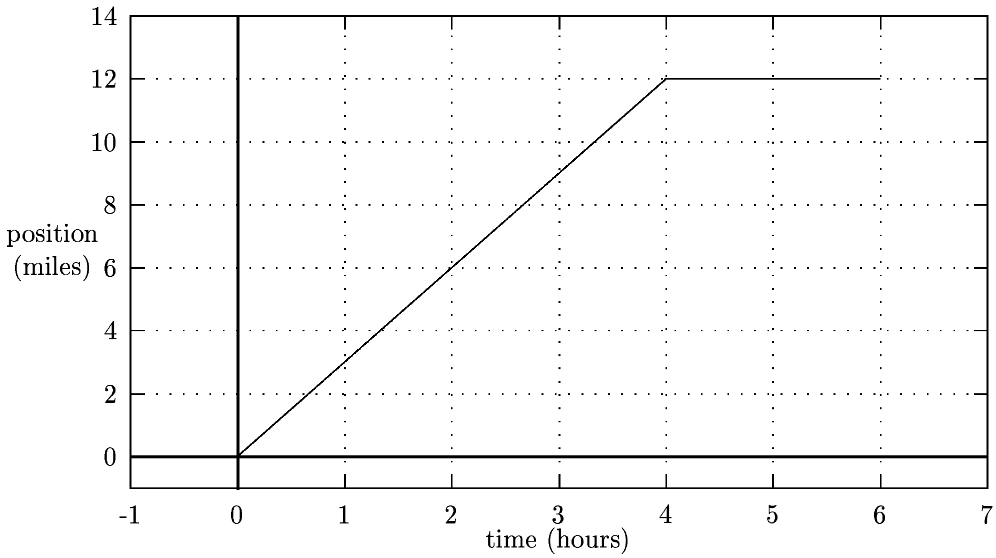


in the picture,

Example 12: Say we want a function to describe the position of a walker who walks at a steady pace of 3 miles per hour for 4 hours, and then stops to take a nap for 2 hours. If we measure the time t in hours and position p in miles from her starting point, a piecewise function describing the position of the walker is

$$p(t) = \begin{cases} 3t & \text{for } 0 \leq t \leq 4 \\ 12 & \text{for } 4 < t \leq 6. \end{cases},$$

The graph of this function is illustrated below.



2.5 More about Domain and Range

Introduction

In precalculus and calculus most functions are given by an algebraic rule, and most such rules give an answer for any real number input.

Conventions and Restrictions for Domains

It is a standard convention that if a function is given by an algebraic rule, the domain is assumed to be the set of all real numbers for which that rule gives meaningful answers. Only two of the basic algebraic operations present obstacles:

- Firstly, since division by zero is impossible, if a formula has a variable in its denominator we must exclude from the domain any number that would give the denominator a value of zero.

  Example 13: Let $f(x) = 3/(x - 2)$. Since substituting 2 for x makes the denominator 0, the domain of the function cannot include 2. Any other substitution for x gives an output. The largest set possible for the domain is the set of all real numbers except 2. (Recall why division by zero is impossible. Any division can be checked by a cross-multiplication; for example, if we want to check that $6/2 = 3$, we multiply $3 * 2$ and get 6. Now, if $6/0$ gives an answer a, then the check is that $0 \times a = 6$. But that's impossible because zero times anything is zero. So no answer checks. So, no division by zero.)

- Secondly, we cannot get a real number for the square root (or any even root) of a negative number. (Recall the reason for this: $\sqrt{a} = b$ means that $b^2 = a$. But squaring either a positive or a negative number b gives a positive answer, not a negative one. For example, there is no number to square to get -4.) So we require that any expression under an even radical be greater than or equal to zero.

  Example 14: Let $g(x) = \sqrt{2x - 10}$. The domain is the set of all x such that $2x - 10 \geq 0$. The solution is the set of all numbers greater than or equal to 5. You may write this $\{x : x \geq 5\}$ or just $x \geq 5$.

Range

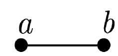
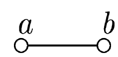
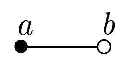
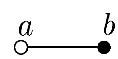
The range of a function given by an algebraic rule consists of what you get as outputs once you know the rule and the domain. It is not always easy to find algebraically. For the functions we work with, it's usually easier to tell from a graph what the outputs are. However, it can be done algebraically in some cases using methods at our disposal.

Example 15: Let $f(x) = 2x + 3$, with domain all real numbers, and suppose we want to find the range. To keep the notation simple, write $y = 2x + 3$. Supposing we have chosen a particular y; how do we tell if there is an x that gives it as an output? We substitute our value for y in the equation $y = 2x + 3$ and solve for x. This gives $x = (y - 3)/2$. For any y we choose, we can get an x, since there are no obstacles to the computations. For example, if $y = 103$, then $x = 50$. The x we get is the input that will give the desired output. Thus we can get any number as an output for f, and the range of f consists of all real numbers.

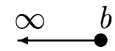
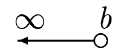
In the preceding example the algebra is simple. It can be difficult or impossible; for example, if $f(x) = x^5 - \sqrt{x}$ you can't get find the range algebraically. In such a case, all is not lost. After introducing *interval notation*, needed for this and other purposes, we will show how to read domains and ranges from graphs. After a little practice, this is often easier than the algebraic method, even if the algebra is possible.

Interval Notation

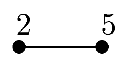
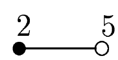
Interval notation is summarized in the tables below:

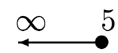
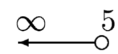
Intervals of Finite Length			
Type of Interval	Notation	Inequality	Graph of Interval
Closed	$[a, b]$	$a \leq x \leq b$	
Open	(a, b)	$a < x < b$	
Half-open	$[a, b)$	$a \leq x < b$	
Half-open	$(a, b]$	$a < x \leq b$	

In addition to these intervals of finite length there are intervals that contain all numbers larger than a given number, or all numbers smaller than a given number. To describe these we use the symbol ∞ (infinity). It is used in interval notation to indicate that all numbers to the right of a given point are included. The symbol $-\infty$ stands for negative infinity, and is used in interval notation to indicate that all numbers to the left of a given point are included.

Infinite Intervals			
Type of Interval	Notation	Inequality	Graph of Interval
Closed	$[a, \infty)$	$a \leq x < \infty$	
Closed	$(-\infty,$ b$]$	$-\infty < x \leq b$	
Open	(a, ∞)	$a < x < \infty$	
Open	$(-\infty, b)$	$-\infty < x < b$	

Examples:

Intervals of Finite Length			
Type of Interval	Notation	Inequality	Graph of Interval
Closed	$[2, 5]$	$2 \leq x \leq 5$	
Open	$(2, 5)$	$2 < x < 5$	
Half-open	$[2, 5)$	$2 \leq x < 5$	
Half-open	$(2, 5]$	$2 < x \leq 5$	

Infinite Intervals			
Type of Interval	Notation	Inequality	Graph of Interval
Closed	[2, ∞)	$2 \leq x < \infty$	
Closed	(- ∞, 5]	$\infty < x \leq 5$	
Open	(2, ∞)	$2 < x < \infty$	
Open	(- ∞, 5)	$\infty < x < 5$	

Note that the number 2 lies in the intervals $[2, 5]$, $[2, 5)$, $[2, \infty)$, and $(-\infty, 2]$, but does not lie in $(2, 5]$, $(2, 5)$, $(2, \infty)$ or $(-\infty, 2)$.

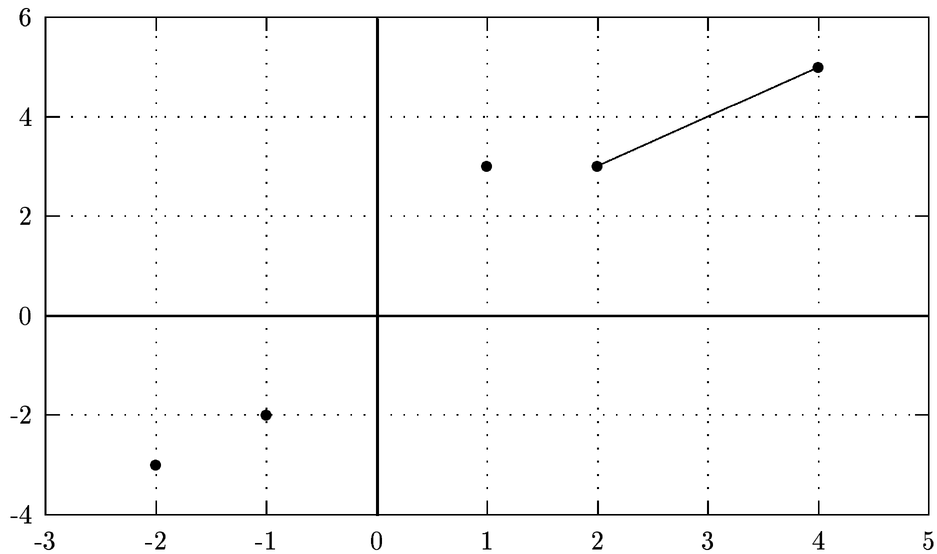
Domain and Range from a Graph

To illustrate the graphical method of finding domains and ranges, we use some rather odd-looking functions. We take the function to be given by the graph and we assume the domain is what you see in the illustration — no extensions beyond the boundary.

The domain of a function given by such a graph is assumed to be the x-values that have outputs shown in the illustration. To see which ones they are, imagine traveling along the x-axis from left to right, looking up and down from each point on the axis to see if you see a point of the graph directly above or below it. The collection of all values of x for which there is an output is the domain of the function.

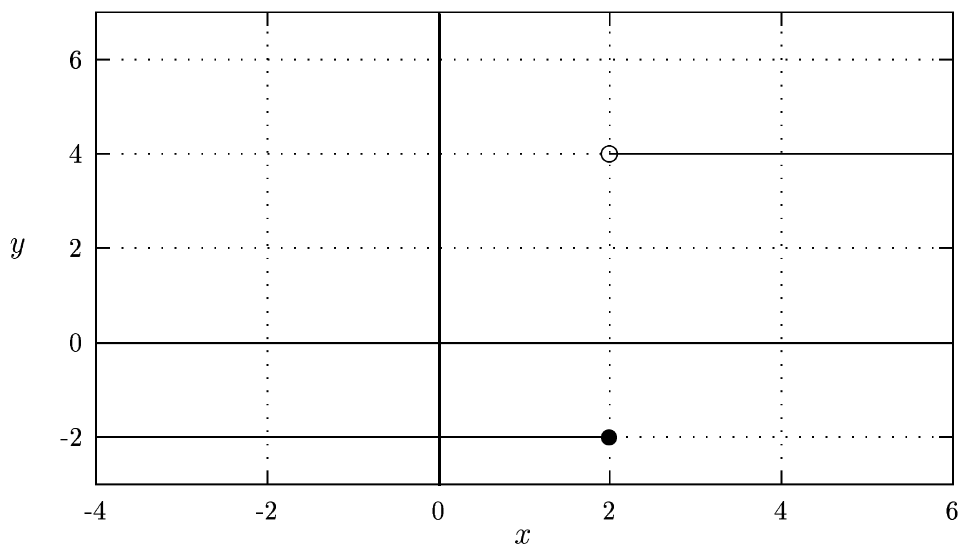
The range consists of all the y values that are outputs. To see which ones they are, imagine climbing the y-axis from bottom to top, looking left and right from each point on the axis in turn for points on the graph. A simple example is the function given in Example 5in the first section of this chapter. The domain is $\{1, 2, 3, 4, 5, 6\}$ and the range is $\{0, 3, 4, 10\}$

Example 16: Consider the following graph of a function.



The x-values for which the function is defined are -2, -1, 1 and all points in interval $[2, 4]$. We use set brackets around the isolated points, so the domain is written $\{-2, -1, 1\} \cup [2, 4]$. (The $\cup$ is read "union" and in this context means "and.") The y-values obtained are -3, -2, 3, and everything from 3 to 5 inclusive. The range is thus $\{-3, -2\} \cup [3, 5]$.

Example 17:



The illustrated domain consists of all values of x between -4 and 6, or in interval notation $[-4, 6]$. The range consists of the two points $y = -2$ and $y = 4$, or in set notation, $\{-2, 4\}$.

2.6 Average Rate of Change

Change and the rate at which it occurs are of central importance in calculus and its applications. The velocity of a probe as it lands on the Martian surface, the amount it costs to produce the million and first widget after you've produced a million, the rate at which the oxygen level is increasing or decreasing in a patient's blood stream — all these are of great importance and all involve rate of change. Calculus deals effectively with such matters using the concept of instantaneous rate of change. Defining change at an instant is a problem, because change happens over time. If the change occurs at a steady rate, it's easy: if Juan the Walker walks steadily and covers 8 miles in 2 hours, his speed at any instant is 4 mph. But if an object is falling toward the surface of Mars it is moving ever faster, and the problem of describing its velocity at an instant is hard. It's been solved by calculus. The method involves taking limits of average velocities over shorter and shorter intervals of time and finding a limit–a number the averages close in on. We leave most of the limits for calculus. Here we develop the idea of average rate of change over an interval.

For a constant function; that is, a function of the type $f(x) = c$, where c is a constant, there is no change, and the rate of change is always zero. This is pretty dull, but worth noting.

For linear functions the rate of change is always the same. It is the slope of the line. Recall the definition of slope: the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

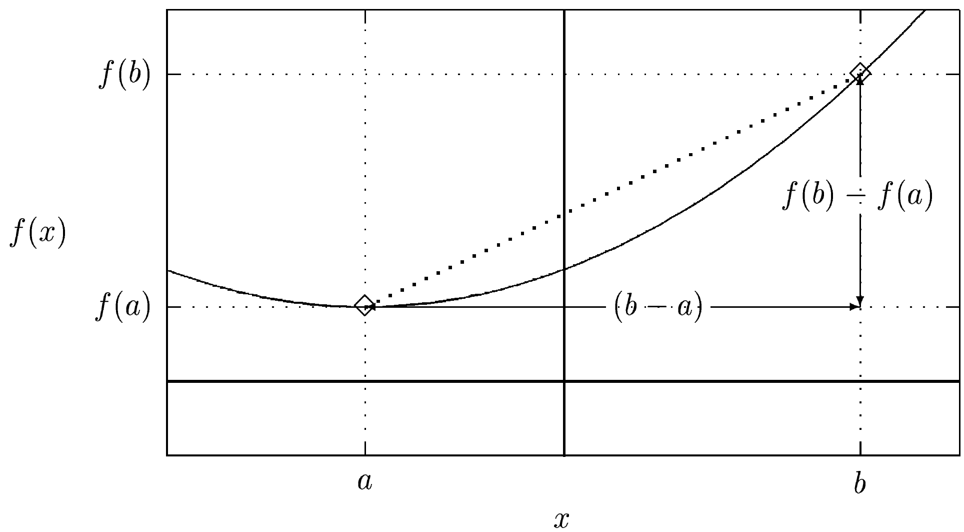
Also recall that no matter what two points of the line are used to find the slope, the result is the same. In terms of the graph of a line in a coordinate system, the slope reflects the fact that the graph rises or falls at a steady rate. Juan the Walker is an example of this.

In many situations in which rate of change is not steady, we can get an overall view by looking at the endpoints of an interval of interest, and finding the average change;; *i.e.* the steady rate that would produce the same result. For example we can take total distance traveled divided by total time taken to find the average velocity of Michael the Meanderer, who hurries then dawdles, or we can take the price of a stock at the beginning and end of a decade and divide by 10 to find its average change per year.

Definition: The *average rate of change* (abbreviated as *aroc*) of a function f between the inputs a and b (or equivalently over the interval $[a, b]$) is given by

$$\text{a.r.o.c.} = \frac{f(b) - f(a)}{b - a}.$$

The resemblance to the slope formula is no coincidence; th aroc is the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$, as is illustrated in the figure below.



f(b)
f(x)
f(a)
f(b) − f(a)
(b − a)
a
b
x