MATH 161 — Precalculus[1]
Community College of Philadelphia

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Math 161 — Chapter 3
Quadratic Functions
Information

3.1 Introduction and Definition of Quadratic Functions

A function which can be written in the algebraic form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are constants $(a \neq 0)$ is called a *quadratic function*. The constants a, b, c are the coefficients. The coefficient of the quadratic term, a, is the *leading coefficient*. This way of writing it is called the expanded form of the equation of the function. In this chapter we study the various properties of quadratic functions.

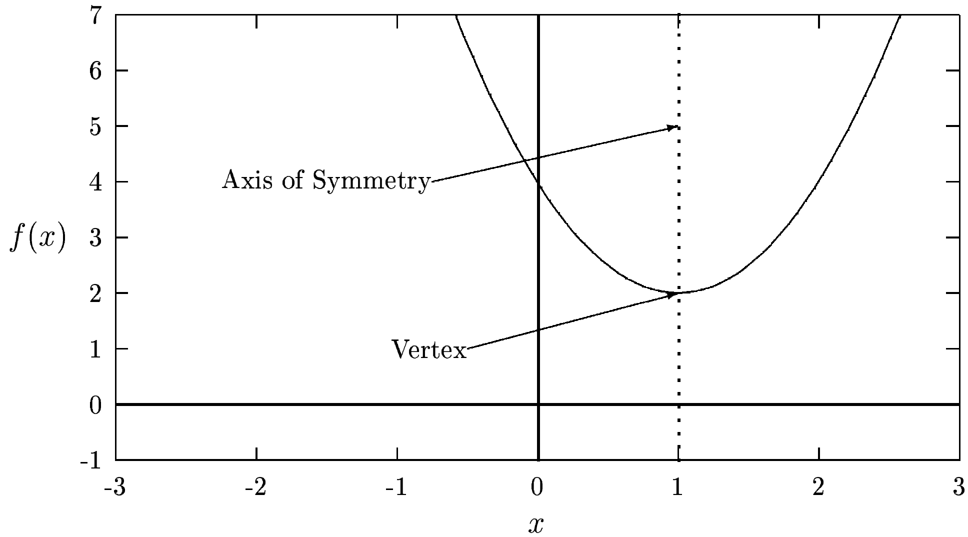
Also, any expression which can be written in the form $ax^2 + bx + c$ where a, b, and c are constants $(a \neq 0)$ is called a *quadratic expression*.

Example 1: If $f(x) = 9x^2 - 36x + 100$, then $f(x)$ is a quadratic function and $a = 9$, $b = -36$, $c = 100$

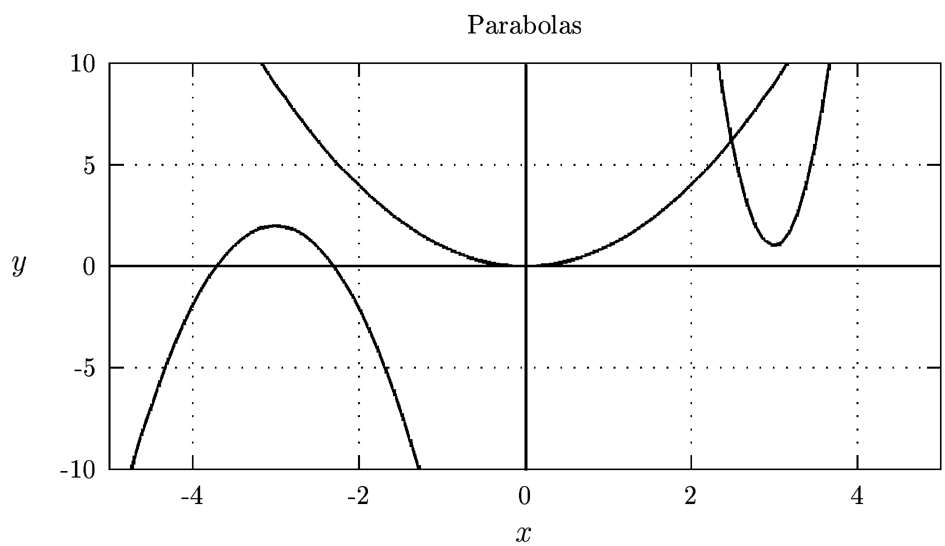
Example 2: If $f(x) = 7 - 8x + 3x^2$, then $f(x)$ is a quadratic function, and $a = 3$, $b = -8$ and $c = 7$.

3.2 Graphs of quadratic functions

The graphs of quadratic functions are *parabolas*. All parabolas have the characteristic shape shown below, although they can differ in size and in the direction in which they open. Every parabola has an *axis of symmetry* – that is, a line that cuts it into two symmetric parts – and a *vertex*, which is the high or low point of its graph, and the point where it cuts its axis of symmetry. The parabola below is symmetric about the dotted vertical line, and its vertex is $(1, 2)$.



The parabolas below show some of the variety possible.



The path of an object thrown in a gravitational field (a ball thrown on planet Earth, for example) is approximately parabolic, though it may be only a small part of the parabola.

3.3 Some Algebra for Quadratic Functions

The x- and y-intercepts and the vertex are points of particular significance on the graph of a quadratic function. They stand out visually. The coordinate values at these points are also important in calculus and applications, and you need to be able to find them from the function formula as well as from the graph. There are different forms of the function formula from which it's easy to read them, and we now indulge in a little algebra in order to derive these forms.

Quadratic Expressions

In algebra you have seen and sometimes factored quadratic expressions; *i.e.*, expressions of the form

$$ax^2 + bx + c.$$

Quadratic expressions may appear in different forms. For example, $x^2+8x+12$ factors into $(x+2)(x+6)$. It is useful to factor to find roots, and it's important to recognize a quadratic as such when it appears in factored form. Not all quadratic expressions factor.

Quadratic Equations

A quadratic equation is one that can be written in the form

$$ax^2 + bx + c = 0.$$

If a quadratic equation appears in some other form, the first thing you should do when working with it is re-write it so that zero is alone on one side of the equation. For example, change $x^2 = -8x - 12$ to $x^2 + 8x + 12 = 0$.

Roots of a Quadratic Equation

The roots of a quadratic equation are the values of the independent variable, (x), that are solutions of the equation. Sometimes these can be found by factorization. For example, the equation

$$x^2 + 8x + 12 = 0$$

can be written in factored form

$$(x+2)(x+6) = 0$$

from which it is clear that

$$x = -2, -6.$$

The roots of this equation are -2 and -6. Substituting them into the equation gives a true result. On the graph of the function

$$f(x) = x^2 + 8x + 12,$$

-2 and -6 are the x-intercepts. (Recall the Principle of Zero Products, which is used in solving the equation above. It says that if a product is zero then at least one of its factors must be zero. That is why we re-write the quadratic expression as a product, and why we want zero on one side of the equation.)

When a quadratic equation does not factor, it may still have roots. To find them we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 3: Let's use the quadratic formula on the equation we just solved by factoring, $x^2 + 8x + 12 = 0$. Here $a = 1$, $b = 8$, $c = 12$, and substituting into the quadratic formula we get

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 12}}{2 \cdot 1} \\ &= \frac{-8 \pm 4}{2} \\ &= \frac{-4}{2}, \frac{-12}{2} \\ &= -2, -6. \end{aligned}$$

We get the same answer as before (as we would certainly hope to do).

Example 4: Now we solve the equation $2x^2 + 8x = 3$ using the quadratic formula. In this case, we first rewrite the equation as $2x^2 + 8x - 3 = 0$, and we have $a = 2$, $b = 8$, $c = -3$. Substituting into the quadratic formula we get

$$\begin{aligned} x &= \frac{-8 \pm \sqrt{8^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2} \\ &= \frac{-8 \pm \sqrt{88}}{4} \end{aligned}$$

$$\begin{aligned} &= -2 \pm \frac{4\sqrt{22}}{4} \\ &= -2 \pm \frac{\sqrt{22}}{2}. \end{aligned} \tag{3.1}$$

Rounding to three decimal places we get

$$\begin{aligned} x &= -2 \pm 2.345 \\ &= -4.345, 0.345. \end{aligned}$$

You should make sure you follow the algebra in this example. Refer to the algebra pushups on simplifying square roots if you have questions.

Note that the quadratic formula does not give a real number root if the quantity under a radical is negative. If the quantity under the radical is zero, then we get one root.

Completing the Square

A process called completing the square is used to change a quadratic function into a form that makes it easy to find the coordinates of the vertex of the function. This process is also used to derive the quadratic formula, and it arises occasionally in calculus as well. You may have seen it in algebra, but here it is in case you didn't, or don't remember it.

To complete the square of a quadratic expression written in the form $ax^2 + bx + c$ means to re-write it in the form $a(x - h)^2 + k$. To achieve this we do the following:

1. Factor the a (the coefficient of x^2) out of the quadratic (x^2) and linear (x) terms. For example,
   - change $3x^2 + 6x + 7$ to $3(x^2 + 2x) + 7$,
   - change $2x^2 - 5x + 8$ to $2\left(x^2 - \frac{5}{2}x\right) + 7$,
   - leave $x^2 + 6x - 3$ as it is.
2. Find half the coefficient of x in the expression that remains inside the parentheses after factoring. For example,
   - in $3(x^2 + 2x) + 7$, take half of 2: $\frac{1}{2} \times 2 = 1$,

- in $2\left(x^2 - \frac{5}{2}x\right) + 7$, take half of $-\frac{5}{2}$: $\frac{1}{2} \times \frac{5}{4} = \frac{5}{4}$,
- in $x^2 + 6x - 3$, take half of 6: $\frac{1}{2} \times 6 = 3$.

3. Square the number found in the preceding step. For example
   - $1^2 = 1$,
   - $\left(-\frac{5}{4}\right)^2 = \frac{25}{16}$,
   - $3^2 = 9$.

4. Add and subtract the square just obtained to the expression inside parentheses. This doesn't change the value of the expression within parentheses because it adds a total of 0 to it. Continuing with our examples,
   - $3(x^2 + 2x) + 7 = 3(x^2 + 2x + 1 - 1) + 7$,
   - $2\left(x^2 - \frac{5}{2}x\right) + 7 = 2\left(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}\right) + 7$,
   - $x^2 + 6x - 3 = x^2 + 6x + 9 - 9 - 3$.

5. Regroup the terms into a perfect square $(x-h)^2$ plus a constant. (Note the use of the distributive law.) Finishing our examples,
   - $3(x^2 + 2x + 1 - 1) + 7 = 3(x^2 + 2x + 1) - 3 + 7 = 3(x + 1)^2 + 4$,
   - $$\begin{aligned} 2\left(x^2 - \left(\frac{5}{2}\right)x + \frac{25}{16} - \frac{25}{16}\right) + 7 &= 2\left(x^2 - \left(\frac{5}{2}\right)x + \frac{25}{16}\right) - \frac{25}{8} + 7 \\ &= 2\left(x - \frac{5}{2}\right)^2 + \frac{31}{8} \\ &= 2\left(x - \frac{5}{2}\right)^2 + 3\frac{7}{8} \end{aligned}$$
   - $x^2 + 6x + 9 - 9 - 3 = (x + 3)^2 - 12$.

6. The process is now complete. Our results are in the desired form:
   - $3(x + 1)^2 + 4$,
   - $2\left(x - \frac{5}{2}\right)^2 + 3\frac{7}{8}$
   - $(x + 3)^2 - 12$.

Example 5:

$$\begin{aligned} x^2 - 10x - 24 &= x^2 - 10x + 25 - 25 - 24 \\ &= x^2 - 10x + 25 + (-25 - 24) \\ &= (x - 5)^2 - 49 \end{aligned}$$

In this example $h = 5$ and $k = -49$.

Example 6:

$$\begin{aligned}
-2x^2 + 5x + 15 &= -2\left[x^2 - \frac{5}{2}x\right] + 15 \\
&= -2\left[x^2 - \frac{5}{2}x + \left(\frac{25}{16} - \frac{25}{16}\right)\right] + 15 \\
&= -2\left[x^2 - \frac{5}{2}x + \frac{25}{16}\right] + (-2)\left(-\frac{25}{16}\right) + 15 \\
&= -2\left[x^2 - \frac{5}{2}x + \frac{25}{16}\right] + \frac{25}{8} + 15 \\
&= -2\left[x - \frac{5}{4}\right]^2 + \frac{145}{8}.
\end{aligned}$$

In this example $h = 5/4$ and $k = 145/8$.

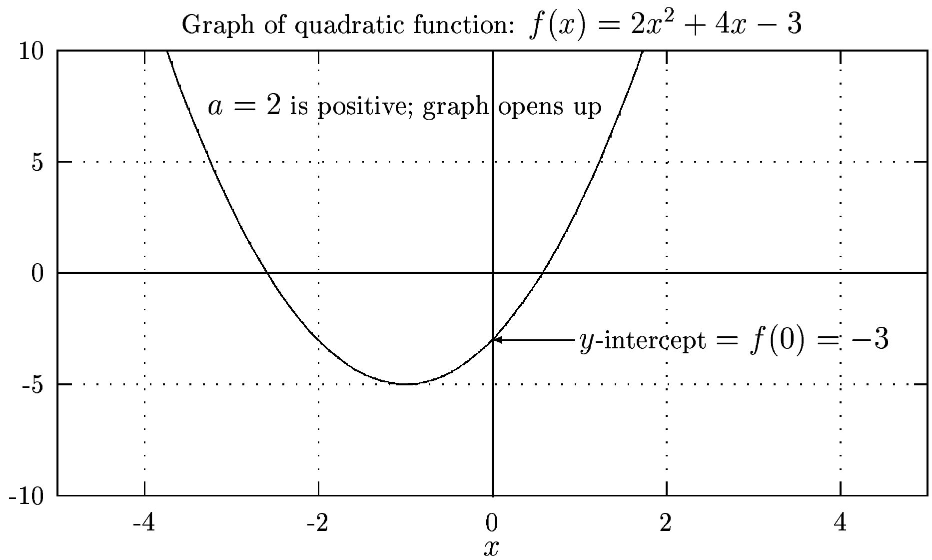
3.4 Forms of a Quadratic Function

Expanded Form

From the *expanded form* of a quadratic function, $f(x) = ax^2 + bx + c$, two of the properties of its graph are easy to read:

1. The y-intercept is $f(0)$, which is c.
2. If the leading coefficient a is positive the graph of the function opens upward and if a is negative it opens downward.

Example 7: The graph of $f(x) = 2x^2 + 4x - 3$ is illustrated below. The leading coefficient is positive so the graph opens upward, and the y-intercept is $c = -3$.



Example 8: Let $f(x) = x^2 + 3x + 2$. The graph of f opens upward and the y -intercept is 2.

Example 9: Let $g(x) = -2x^2 + 5x - 3$. The graph of g opens downward and the y-intercept is -3.

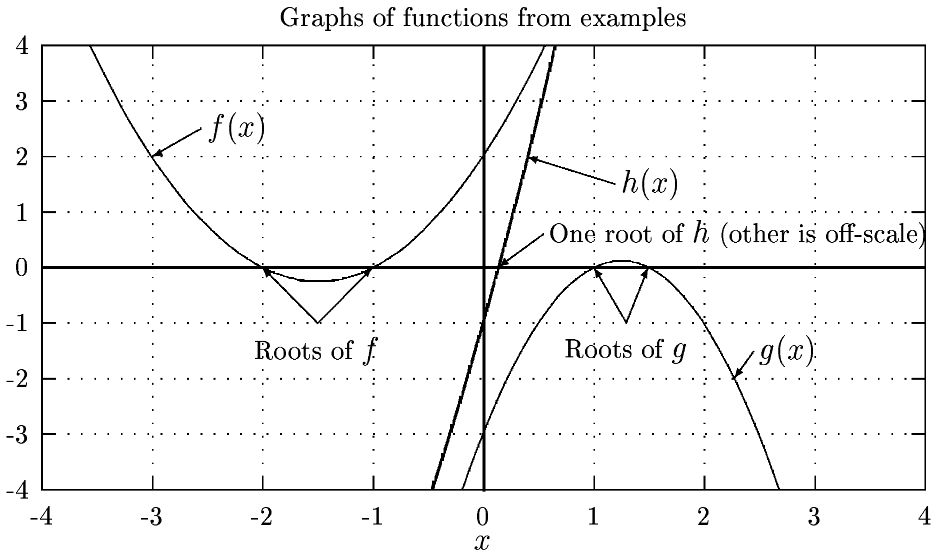
Factored Form

From the factored form of a quadratic function, $f(x) = a(x - r_1)(x - r_2)$, we can immediately read the roots of the function: they are r_1 and r_2.

Example 10: Let $f(x) = x^2 + 3x + 2 = (x + 1)(x + 2)$. The roots of f are -1 and -2.

Example 11: Let $g(x) = -2x^2 + 5x - 3 = -(2x - 3)(-x - 1)$. The roots of g are 1 and $\frac{3}{2}$.

Example 12: Let $j(x) = x^2 + 7x - 1$. This does not factor in the usual way, and we use the quadratic formula to find that the roots are $\frac{-7 \pm \sqrt{53}}{2}$, or approximately 0.14 and -7.14. In fact, after finding the roots we could factor the quadratic as $j(x) = \left(x - \left(\frac{-7+\sqrt{53}}{2}\right)\right)\left(x - \left(\frac{-7-\sqrt{53}}{2}\right)\right)$. There is no particular reason to do this except to emphasize the correspondence between roots and factors: if r is a root then $(x - r)$ is a factor and vice versa. But when we say a function is factorable we mean, unless otherwise stated, that the factors have integer coefficients, so we wouldn't call $j(x)$ factorable.



Note from the graphs that the roots are the x-intercepts. The roots of j need be found to only one decimal place for purposes of graphing, since better approximation won't show up in the graph (at least not in this graph).

Standard Form

The standard form of a quadratic function is $f(x) = a(x-h)^2 + k$. Note that $f(h) = k$. If a is positive then k is the smallest value the function takes, since $(x-h)^2$ is never less than zero, so adding $a(x-h)^2$ to k always gives a result greater than or equal to k. If a is negative then k is the largest value the function takes, since $a(x-h)^2$ is never greater than zero, so adding $a(x-h)^2$ to k always gives a result smaller than k.

Whether the graph of $f(x) = a(x-h)^2 + k$ is a parabola opening upward or downward, the point (h, k) is its vertex — the highest or lowest point on the parabola. From the standard form of a quadratic $f(x) = a(x-h)^2 + k$, then, we can immediately read the coordinates of its vertex: (h, k). To put a quadratic function in standard form if it is given in expanded form, complete the square. If it is given in factored form, put it in expanded form first (unless it happens to be a perfect square).

Example 13: Let $f(x) = (x-1)^2 + 7$. Here $f(1) = 7$, and 7 is the smallest value that f ever takes.

Example 14: Let $g(x) = -2(x-4)^2 + 5$. Here $g(4) = 5$, and this is the largest value the function takes.

Summary

We have three forms for quadratic functions, each of which is useful for a particular purpose:

- Expanded form: $f(x) = ax^2 + bx + c$. The y-intercept is c, and the sign of a indicates whether the graph opens up or down,
- Standard form: $f(x) = a(x - h)^2 + k$. The vertex of the parabola is (h, k),
- Factored form: $f(x) = a(x - r_1)(x - r_2)$. The roots are r_1 and r_2.

We have described how to change a function from one form to another.

Example 15: If $f(x) = (x - 8)(x + 3)$ then $f(8) = 0$ and $f(-3) = 0$, and the roots are 8 and -3.

Example 16: Suppose $f(x) = (2x - 9)(x + \pi)$. This can be re-written as $f(x) = 2(x - \frac{9}{2})(x - (-\pi))$, and the roots are $\frac{9}{2}$ and $-\pi$.

Example 17: To change $f(x) = x^2 - 5x + 6$ to factored form we just factor: $f(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$.

Example 18: We may have a quadratic that doesn't factor in the usual way but can still be written in factored form. For example, $f(x) = x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. In algebra the expression $x^2 - 3$ is regarded as unfactorable, but we see here that if we broaden the rules a but, we can put it in factored form.

Example 19: The quadratic $g(x) = x^2 + 4$ doesn't factor. (Here too the rules can be broadened to make factorization possible, but this requires the introduction of new, non-real numbers, called complex numbers. They will be introduced later.) We cannot write it in factored form with real r_1 and r_2.