MATH 161 — Precalculus[1]
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Math 161 — Chapter 5
Square Root and Absolute Value Function; Translations and Reflections
Information

5.1 Introduction

In this chapter we introduce two standard functions that have properties not seen in either linear or quadratic functions: the square root function and the absolute value function. We then use them as examples for introducing some operations that are performed on functions: shifts and reflections. The new functions are connected to functions we have already worked with, and the new operations enable us to establish some relationships among different functions. So we are continuing what we have been doing: analyzing functions by type and examining variations within each type.

5.2 Square root function

Square roots

Before we define the square root function, we recall the definition of square root and give some examples.

Definition: A *square root* of a number N is a number a with the property that $a^2 = N$. We write $a = \sqrt{N}$.

Example 1: Since $3^2 = 9$, $3 = \sqrt{9}$. Note that since $(-3)^2 = 9$, -3 is also a square root of 9.

Example 2: There is no integer or fraction we can square to get 2. (This can be proved.) However, we can get close approximations. For example, $1.4^2 = 1.96$, which is less than 2, and $1.5^2 = 2.25$, which is greater than 2. So if there were a number a such that $a^2 = 2$, a would be between 1.4 and 1.5. If we want to do better, we can experiment to find that $1.41^2 = 1.9881$, and $1.42^2 = 2.0164$. So a is between 1.41 and 1.42. We could continue in this manner, and find numbers whose squares get closer and closer to 2.

We could say that 2 doesn't have a square root, and all we can get are approximations, or we can say that there are numbers that can't be represented by fractions or integers, but have decimal approximations to any degree of accuracy we want. The human race has chosen the latter route. To many it seems mystical, but we say that there is a number $\sqrt{2}$. However, this number is *irrational*; that is, it is not a fraction or integer.

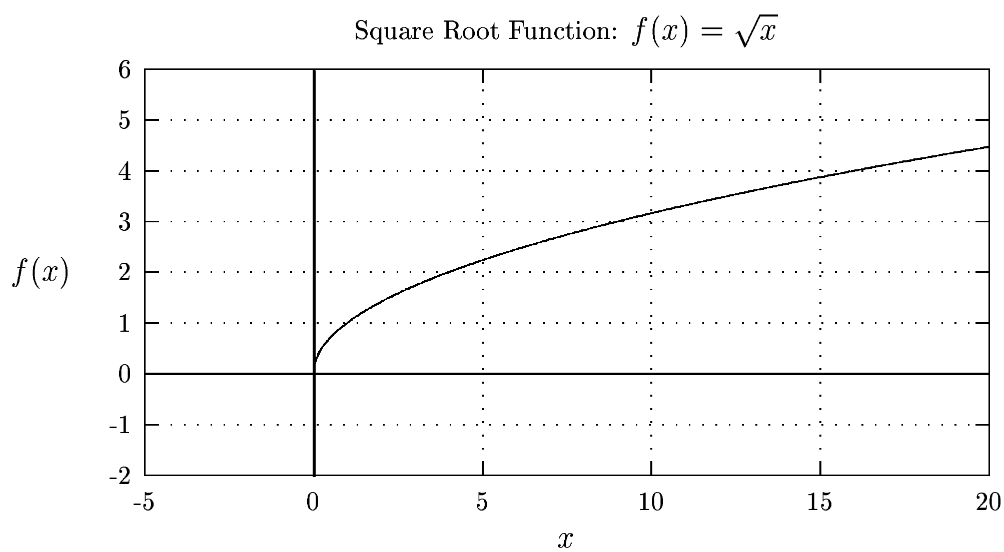
Example 3: $\sqrt{0} = 0$

Example 4: The number -1 does not have a real number square root, since there is no number a such that $a^2 = -1$. New (even more mystical) numbers are invented to take care of this, too, but for our present purposes, we say that $\sqrt{-1}$ is not a real number, so there is no solution to the equation $a^2 = -1$.

It is a convention to assume that the symbol $\sqrt{N}$ stands for the positive square root of N. If you mean to indicate the negative root, write $-\sqrt{N}$.

The Square Root Function

We define the *square root function* to be $f(x) = \sqrt{x}$. Its graph is shown below.



Unlike linear and quadratic functions, the square root function does not have all real numbers in its domain. We cannot allow negative numbers, because they do not give real number answers. But all positive numbers and zero do. The domain of $f(x) = \sqrt{x}$ is therefore $[0, \infty)$. The only numbers we can get as answers are the positive numbers and zero, so the range of

$f(x) = \sqrt{x}$ is also $[0, \infty)$.

Notice that the graph is half of a sideways parabola. Replacing

$$f(x) = \sqrt{x}$$

by

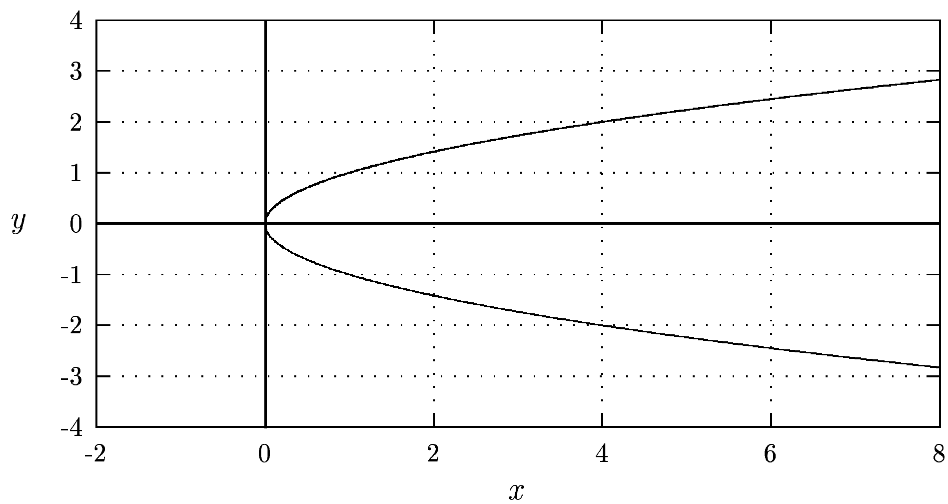
$$y = \sqrt{x}$$

for convenience, and squaring both sides, we get

$$y^2 = x.$$

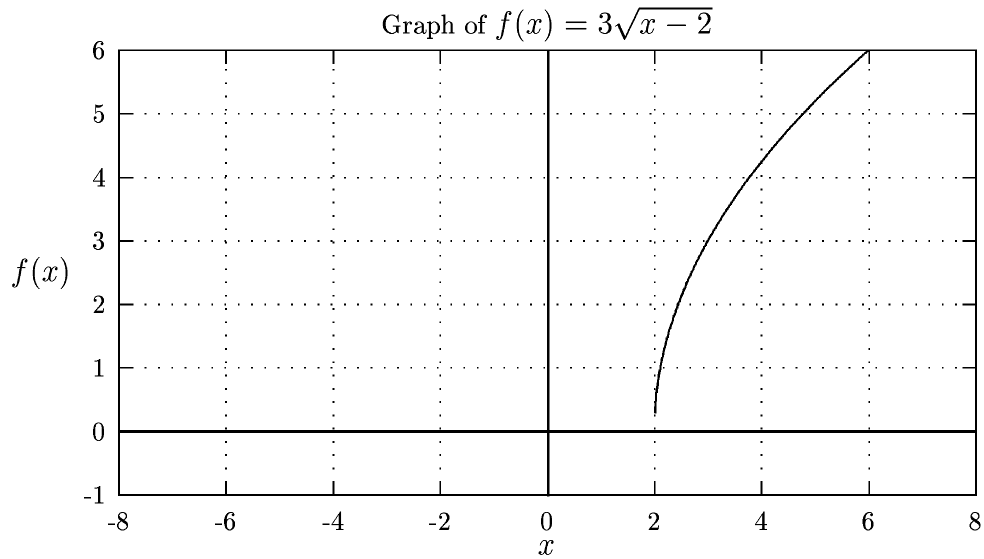
This is the equation of the basic parabola $y = x^2$, with x and y interchanged. A graph of this equation is shown below.

Graph of $y^2 = x$

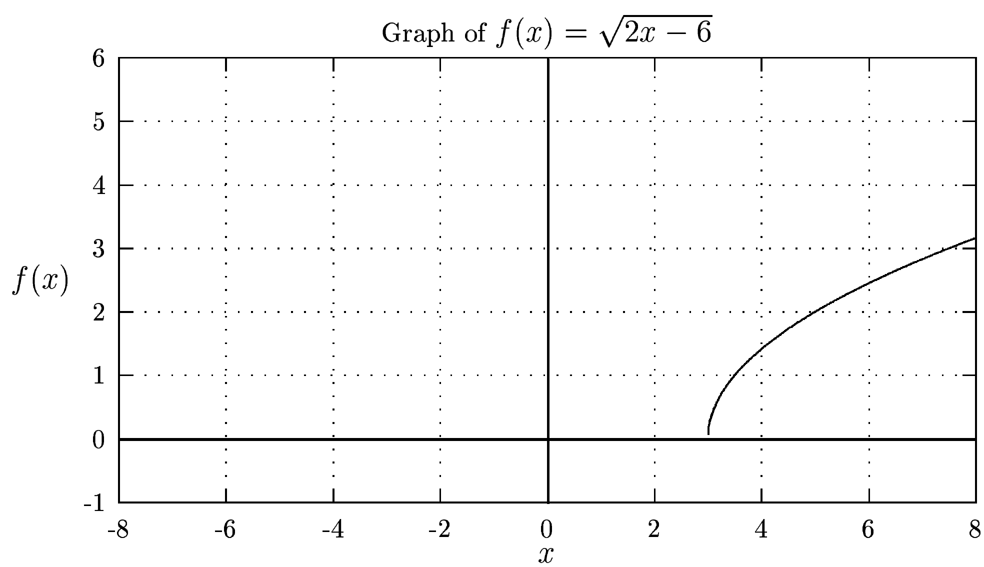


Of course, $y^2 = x$ does not give y as a function of x, as we can see by the graph. Note that the upper branch (half) of the graph is the graph of the function $f(x) = \sqrt{x}$. The lower branch is the graph of the function $g(x) = -\sqrt{x}$. Functions of the form $f(x) = \sqrt{x + c}$, where c is a constant, have very similar graphs. So does any function of the form $f(x) = a\sqrt{bx + c} + d$, where a, b, c, d are any constants. The domain of any of these other square root functions depends on b and c: we must have $bx + c \geq 0$, because the quantity under the radical must be greater than or equal to zero. We will refer to all such functions as square root functions; *the* square root function is $f(x) = \sqrt{x}$

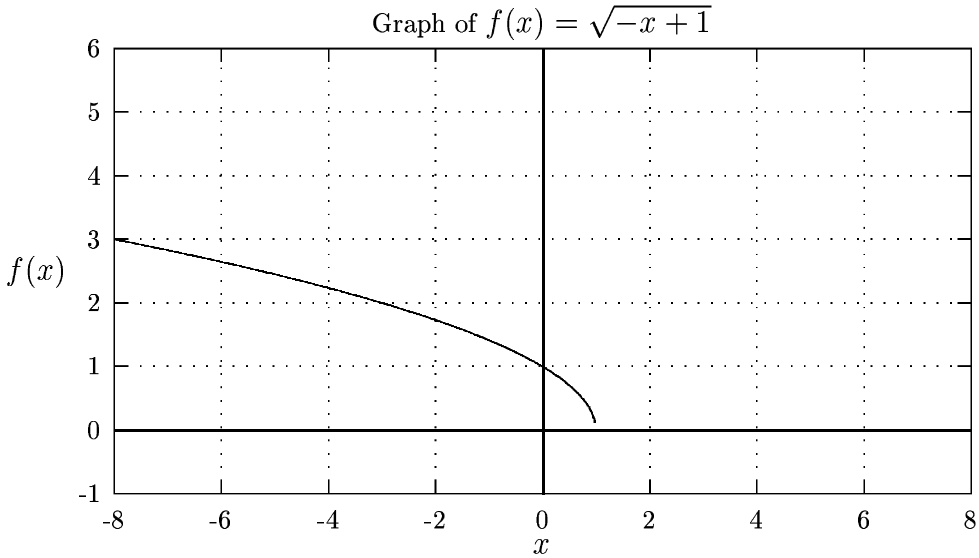
Example 5: Let $f(x) = 2\sqrt{x-5}$. Since $x - 5$ must be greater or equal to 0, the domain of this function consists of all numbers greater than or equal to 5. The range is $[0, \infty)$. The graph is shown below.



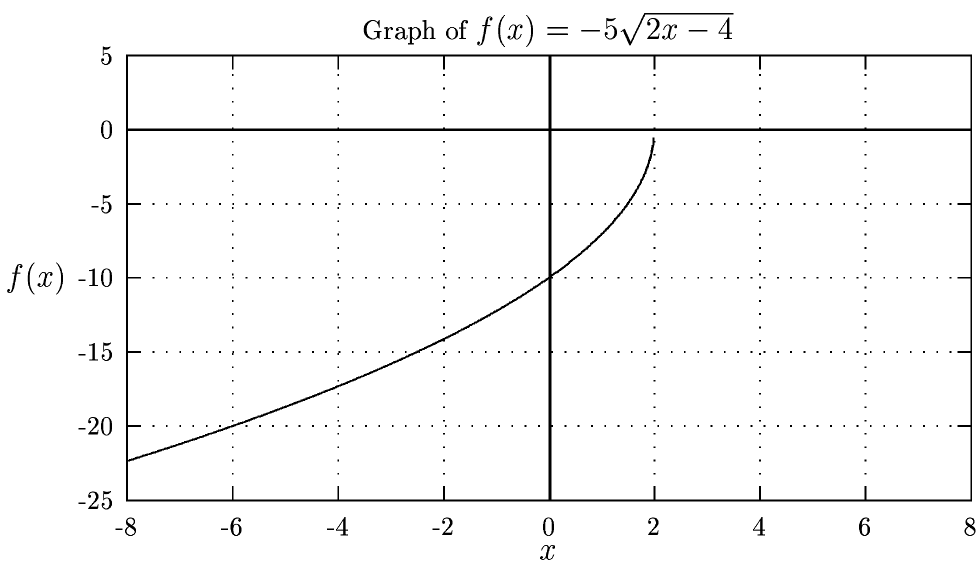
Example 6: Let $f(x) = \sqrt{2x-6}$. Since $2x - 6$ must be greater than or equal to 0, the domain of this function consists of all numbers greater than or equal to 3. The range is $[0, \infty)$. The graph is shown below on the same scale as the graph in the previous example.



Example 7: Let $f(x) = \sqrt{-x+1}$. The domain of this function consists of all numbers $x \leq 1$ The range is $0, \infty)$. The graph is shown below on the same scale as the graph in the previous examples.



Example 8: Let $f(x) = -5\sqrt{2x-4}$. The domain of this function is $(-\infty, 2]$. The graph is shown below.



5.3 Absolute value function

Absolute value

The usual way of thinking of the *absolute value* of a number is as the "positive" of the number; practically speaking, we just drop the negative sign, if there is one. This way of thinking doesn't work well when we deal with variables that may take either positive or negative values. We have to find a mathematical way to describe what you do with a negative quantity to get its "positive." What mathematical operation can we perform on 3 to get -3, for example? We just take the negative of it: -(-3)=3. We use this observation to define absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{5.1}$$

The absolute value of a number can be thought of as its distance from 0 on the number line, and the absolute value of the difference of two numbers is the distance between them on the number line.

Example 9:

$$|7 - 3| = 4 = |3 - 7|$$

The distance between 3 and 7, regarded as points on the number line, is 4. Note the order of the numbers doesn't matter.

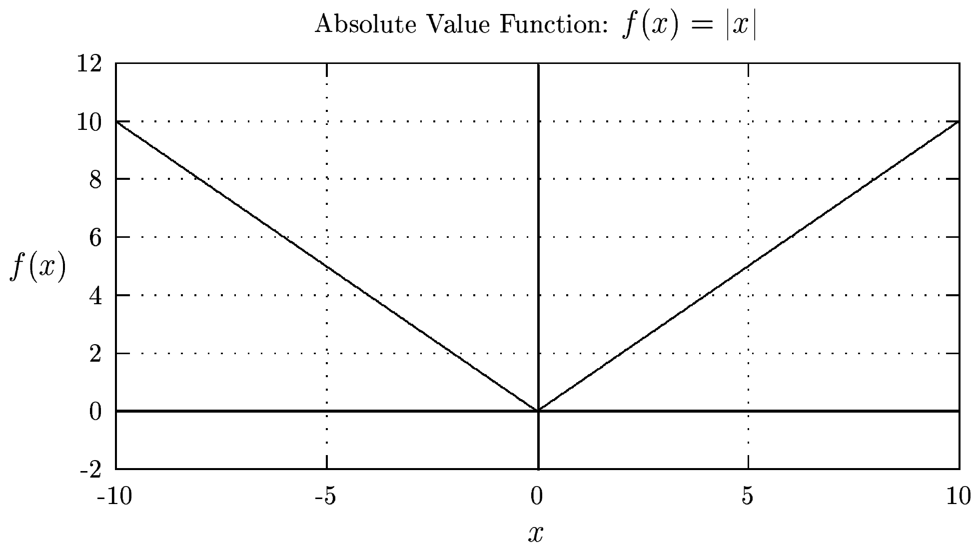
Example 10:

$$|7 - (-3)| = 10 = |-3 - 7|$$

Thus 10 is the distance between the points -3 and 7 on the number line.

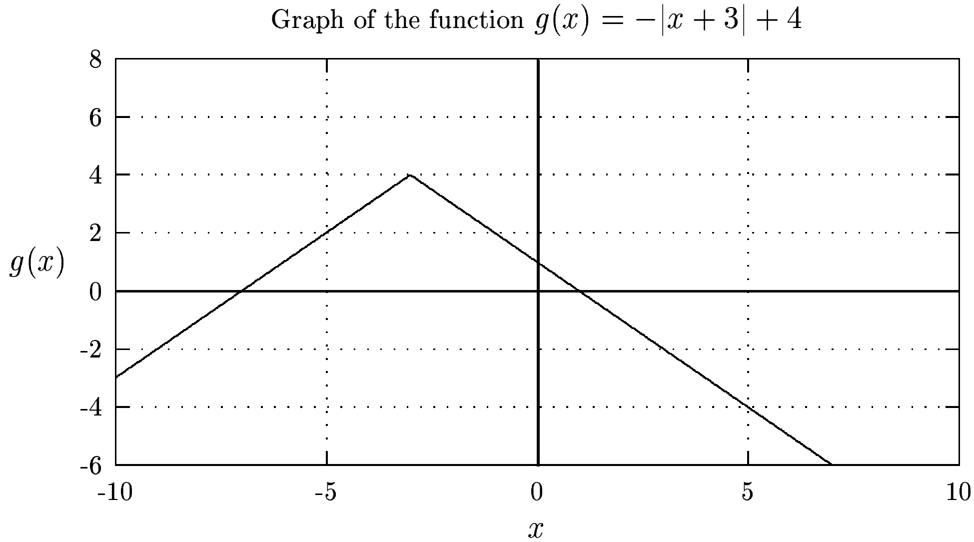
The Absolute value function

We define the absolute value function to be $f(x) = |x|$. This is a particularly easy function to evaluate for any given number; you can do it in your head. The absolute value function is piecewise defined, and its graph has a corner (vertex) at the origin:



Any real number has an absolute value, so the domain of the absolute value function is all real numbers. The range is all real numbers $y \geq 0$.

As with the square root function, there are variations on the absolute values function. Any function of the form $f(x) = a|bx + c| + d$ has similar properties. We will refer to all such functions as absolute value functions. Their graphs will have a similar shape, their domains will be all real numbers, and their ranges will be all numbers greater than the y-value of the vertex, if the graph opens upward, like the graph of $f(x) = |x|$. If a is negative, the graph opens downward, and the range is all y less than or equal to the y-value at the vertex. For an example, see the graph of the the function $g(x) = -|x + 3| + 4$ below.



In this example, we see from the graph that the range is $(-\infty, 4)$. The domain is, as ever for functions of this type, all real numbers.

5.4 Translations and reflections

We have mentioned variations on the basic square root and absolute value functions: we started with $f(x) = \sqrt{x}$ and $g(x) = |x|$ a and observed that functions of the type $h(x) = ax\sqrt{bx+c} + d$ and $j(x) = ax|bx+c| + d$ have similar properties and similar graphs. We could have considered linear and quadratic functions in the same light, starting with $f(x) = x$ and $g(x) = x^2$ and regarding functions of the forms $h(x) = mx + b$ and $j(x) = ax^2 + bx + c$ as variations on these. In fact, we could take any function and modify it in a similar way. The resulting function would bear a family resemblance to the original one. In this section we analyze some of these modifications and the relationships of the new functions to the old.

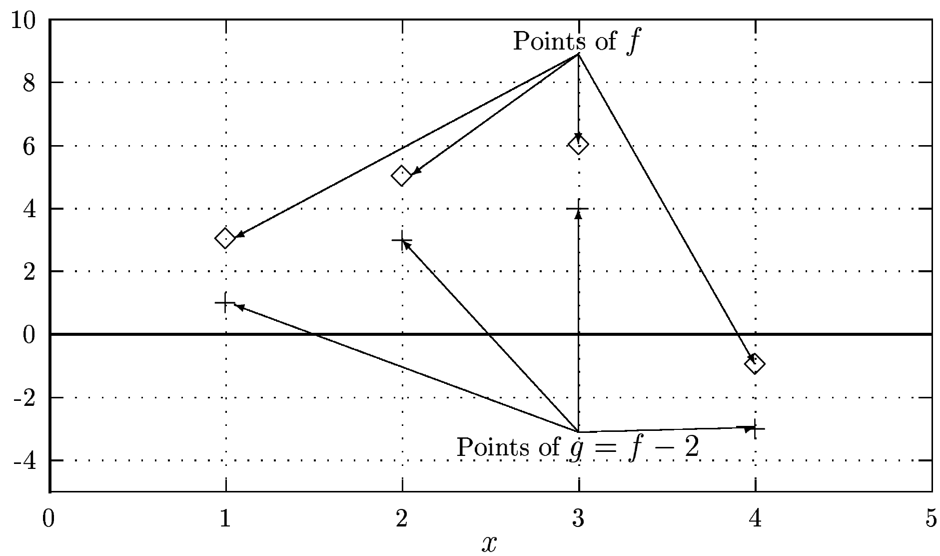
Translations

We start with *translations*. Graphically, a translation (also called a *shift* $g(x)$ of a function $f(x)$ is a function whose graph is exactly the same size and shape as the graph of $f(x)$ but moved horizontally and/or vertically.

Vertical translations

If a point (a, b) is on the graph of a function f and $g(x) = f(x) + d$, then the point $(a, b + d)$ lies on the graph of g. Foe example, even without knowing the formula for $f(x)$, we know that if the point $(7, 2)$ lies on the graph of f, then the point $(7, 6)$ lies on the graph of $g(x) = f(x) + 4$. If a function f is given by a graph, we can graph $g(x) = f(x) + c$ just by adding the value c to each y-coordinate.

Example 11: The graph below shows a function f consisting of the points $(1, 3)$, $(2, 5)$, $(3, 6)$ and $(4, -1)$, and the function $g(x) = f(x) - 2$ consists of the points $(1, 1)$, $(2, 3)$, $(3, 4)$ and $(4, -3)$. Observe that when the graph of f is given, we can graph $g(x) = f(x) + d$ just by adding d to the y-coordinate of each point of f.



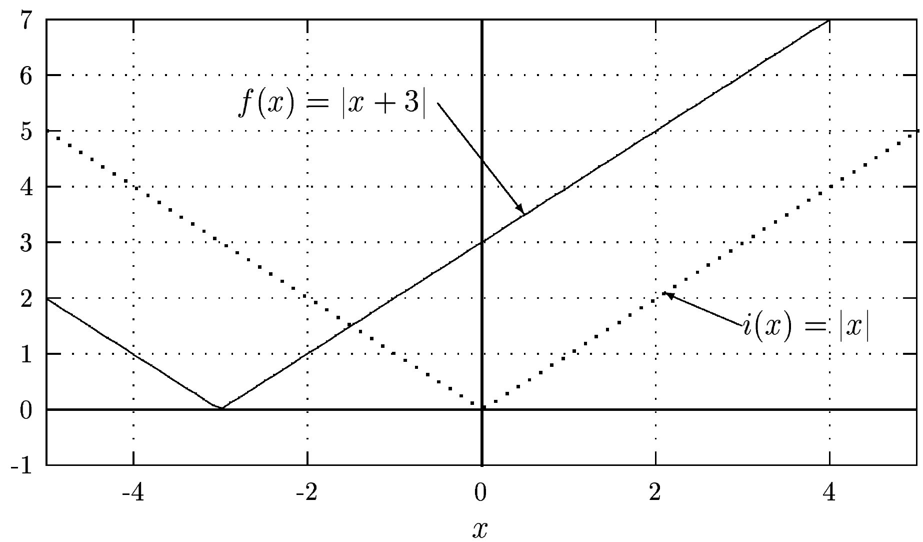
A vertical translation of a function given by an algebraic formula can be sketched similarly: get some points on the new graph by plotting the translations of a few of the points on the original graph, and connecting the points.

Horizontal translations

If the graph of a function f is shifted two units to the right to produce the graph of a function g, and the point $(5, 4)$, for example lies on the graph of f, then the point $(7, 4)$ lies on the graph of g. So, in this case, $g(7) =$

$f(5) = 4$. **In general, when g is obtained from f by a shift of c units horizontally**, $g(x + c) = f(x)$. It is customary to write this as $g(x) = f(x - c)$, which amounts to the same thing.

Example 12: Let $f(x) = |x + 3|$. The graph of $f(x)$ is shown below, with the graph of $i(x) = |x|$.

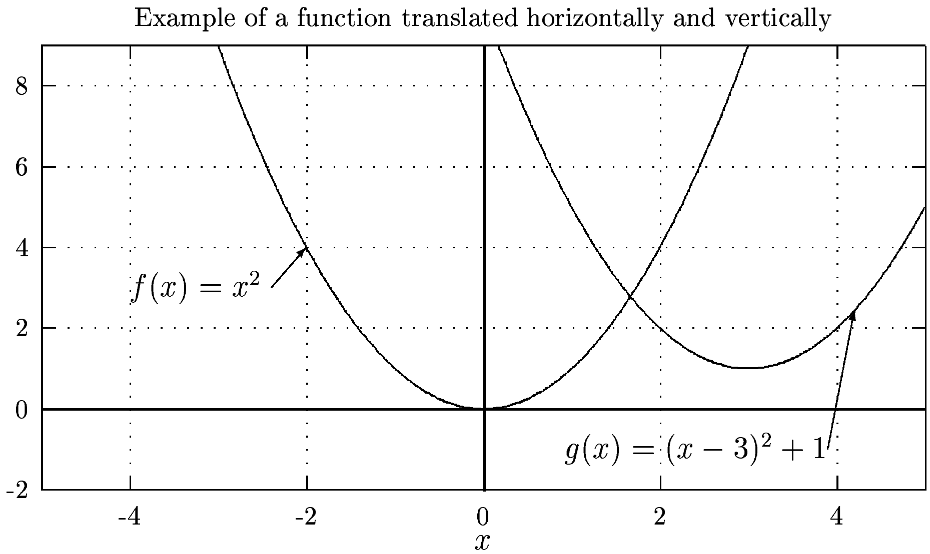


The graph of $f(x)$ is that of $i(x) = |x|$ shifted 3 units to the left. This is parallel to what we have seen with quadratics: the graph of $j(x) = (x + 3)^2$ has its vertex at $(-3, 0)$, and is the graph of $k(x) = x^2$ shifted three units to the left.

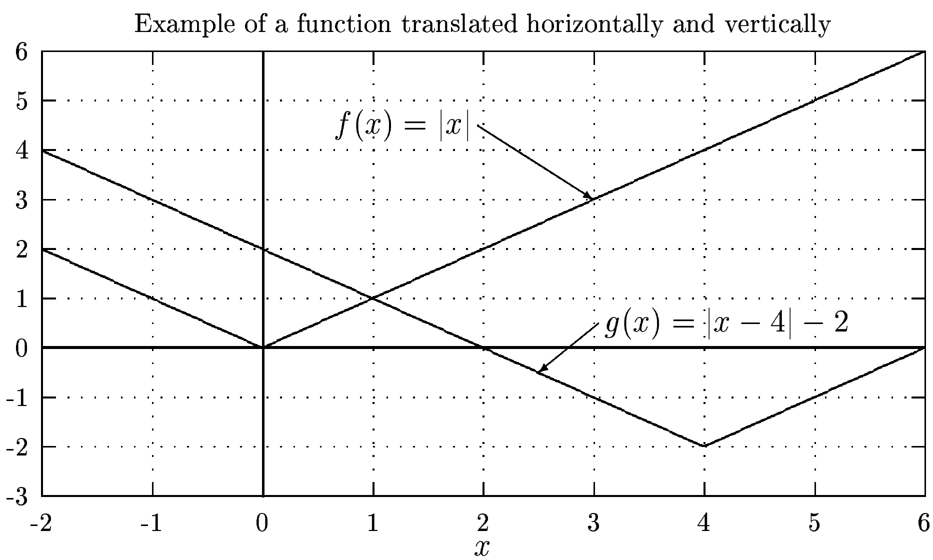
General translations

If a function g is obtained from a function f by translating f d units vertically and c units horizontally, then$g(x) = f(x - c) + d$.

Example 13: The function $g(x) = (x - 3)^2 + 1$ is the translation of the function $f(x) = x^2$ by 3 units to the right and 1 unit up.The domain of both functions consists of all real numbers. The range of f is $[0, infty)$ and the range of g is $[1, infty)$. Note the vertex of f is $(0, 0)$ and the vertex of g is $(3, 1)$.



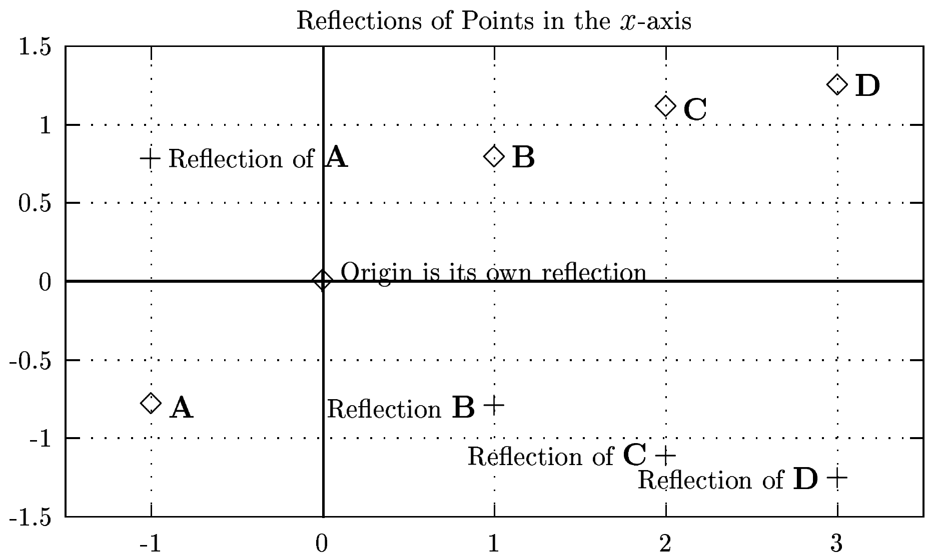
Example 14: The function $g(x) = \sqrt{x+2} - 5$ is the translation of $f(x) = \sqrt{x}$ by 2 units to the left (since $(x+2) = (x-(-2))$ and 5 units down. The domain of f is $[0, \infty)$ and so is the range. The domain of g is $[-2, \infty)$ and the range is $[-5, \infty)$.



The function $g(x) = |x-4| - 2$ is the translation of $f(x) = |x|$ by 4 units to the right and 2 units down. The domain of both functions is $(-\infty, \infty)$; the range of f is $[0, \infty)$ and that of g $[-2, \infty)$

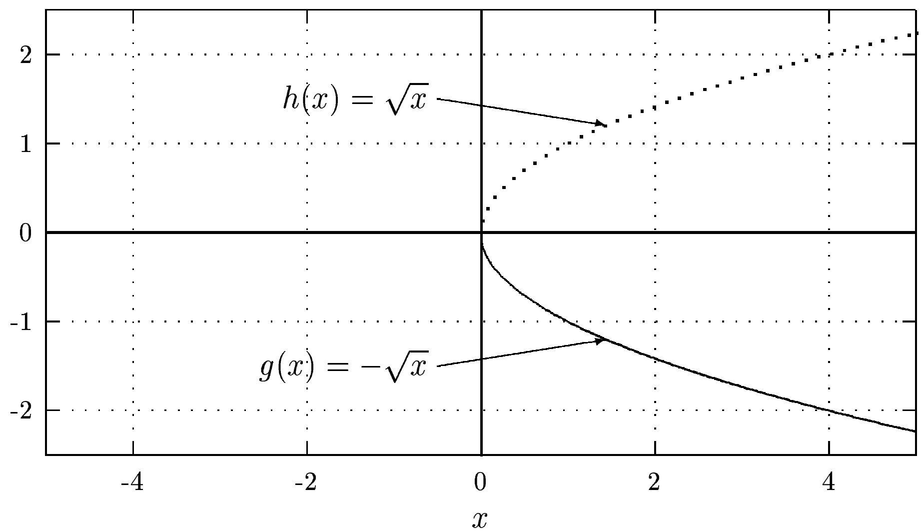
Reflections

Your reflection in a mirror appears to be as far behind the mirror as you are in front of it, and straight back from where you are. In the two-dimensional plane, the*reflection* of a point P in a line l is defined in a way that conforms with this description. The reflection of P in l is the point P' that is as far from l as P is, on the opposite side. The line segment between P and P' is perpendicular to l and is cut in half by l. The reflection of a graph is defined in the same way: the reflection of the graph consists of the reflections of the points of the original graph.



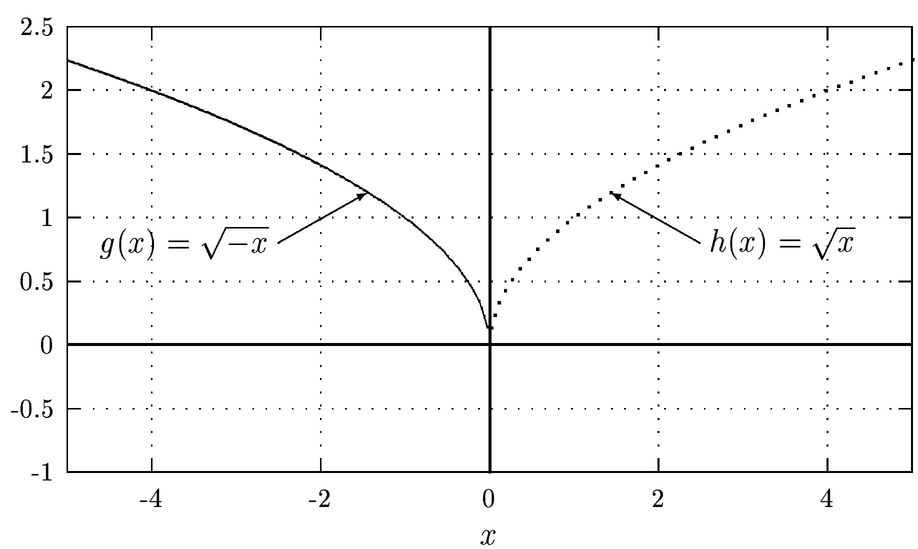
The coordinates of the reflection of a point P are particularly easy to give if the reflecting line is the x- or y-axis. The reflection of a point (x, y) in the x-axis is the point $(x, -y)$ and its reflection in the y-axis is the point $(x, -y)$. For example, the reflection of $(2, 3)$ in the x-axis is$(2, -3)$ and its reflection in the y-axis is $(-2, 3)$.

Examine the function $g(x) = -\sqrt{x}$ shown here on the same coordinate system with $h(x) = \sqrt{x}$. It is a reflection of $h(x)$ in the x-axis.



A reflection of any function f in the x-axis is constructed the same way: **The graph of $-f(x)$ is the reflection in the x-axis of the graph of $f(x)$.**

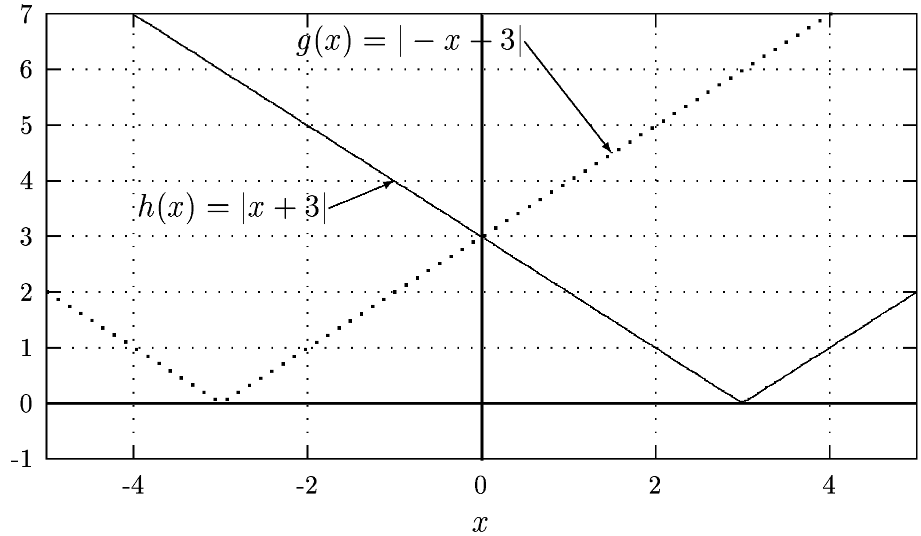
Now look at the graph of the function $g(x) = \sqrt{-x}$ on the same coordinate system with the graph of $h(x) = \sqrt{x}$.



These two graphs are reflections in the y-axis. **For any function f, the graph of $f(-x)$ is the reflection in the y-axis of the graph of $f(x)$.**

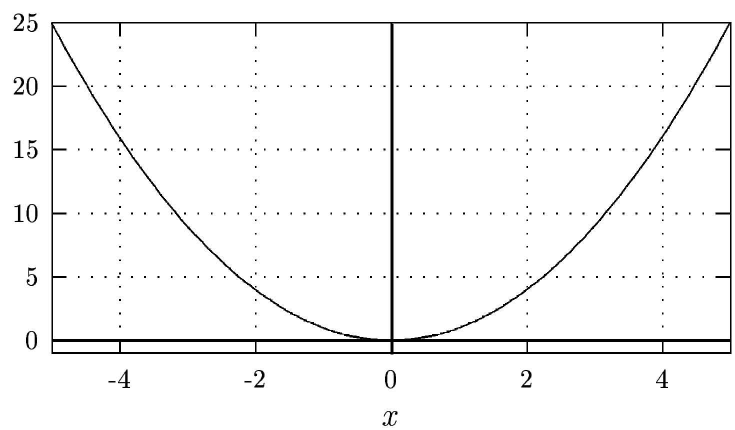
The square-root functions illustrated in the previous paragraphs have limited domains. Functions that are defined for all x also can be reflected.

For example, the reflection about the y-axis of the function $f(x) = |x+3|$ is the function $h(x) = |-x+3|$:



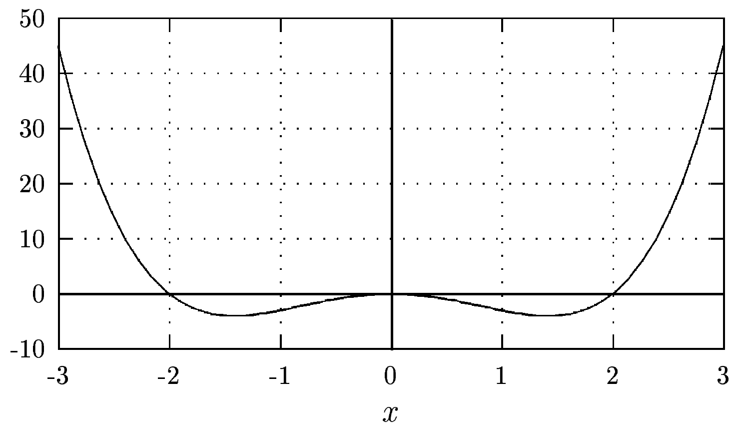
5.5 Symmetry

Consider the graph of the function $f(x) = x^2$:



Examination of this graph should make it clear that the reflection of f about the y-axis lies exactly on top of the graph of f. In other words, the graph of f is its own reflection. We say that the function f is *symmetric about the y-axis.* (It should be clear that f is *not* symmetric about the x-axis.)

A figure in the plane is *symmetric* about a line in the plane if the line acts as a mirror in which half the figure is the reflection of the other half. This means that for every point P on the figure there is another point Q on the figure, on the other side of the line and just as far from the line as P is. The figure below shows the graph of another function that is symmetric about the y-axis.



Note that if the graph were folded over the y-axis the right and left sides of it would match.

Any line can serve as an *axis of symmetry* (that is, the line about which a figure is symmetric), but the lines of greatest interest are the x- and y-axes. It is comparatively easy to tell from an equation whether or not the graph of the equation has symmetry about either of these axes. If it does we can use the fact to reduce our work: we graph half the curve, then an easily graph its reflection.

Symmetry about the y-axis

If a graph is symmetric about the y-axis, then if a point (x, y) is on the graph, so is $(-x, y)$. These two points are at an equal distance from the y-axis, and on opposite sides of it. Thus if the points are on the graph of a function f, we have $f(-x) = f(x)$.

Example 15: The graph of the function $f(x) = x^2$ is symmetric about the y-axis. We can see this from the graph, and more importantly, since the accuracy of graphs is limited, we can see it from the algebra: $f(-x) =$

$(-x)^2 = x^2 = f(x)$.

More generally, we may be interested in a relation between x and y given by an equation that does not necessarily give y as a function of x.

Example 16: Consider the equation

$$2x^2 + y^2 = 2.$$

If we replace x by $-x$, we get the equation

$$2(-x)^2 + y^2 = 2.$$

Since $(-x)^2 = x^2$, the equation $2(-x)^2 + y^2 = 2$ becomes

$$2x^2 + y^2 = 2.$$

Thus replacing x by $-x$ gives a new equation equivalent to the old one. This is the algebraic test for symmetry about the y-axis.

Definition: The graph of an equation is *symmetric with respect to the y-axis* if replacing x by $-x$ in the equation gives an equivalent equation. (Recall that two equations are equivalent if they have exactly the same solution set, and if they are equivalent either can be transformed into the other by the standard simplification techniques, such as replacing $(-x)^2$ by x^2, multiplying both sides of the equation by -1, *etc.*)

Example 17: To test the equation

$$3x^2y + 2x^4y^3 - 8 = 0$$

for symmetry with respect to the y-axis, replace x by $-x$ every place it appears. This gives

$$3(-x)^2y + 2(-x)^4y^3 - 8 = 0$$

which can be simplified to

$$3x^2y + 2x^4y^3 - 8 = 0$$

which is the original equation. This tells us that the graph of the equation is symmetric about the y-axis.

Example 18: To test the equation

$$5x^2 - 4x^3y = 3$$

for symmetry with respect to the y-axis, replace x by $-x$ every place it appears. This gives

$$5(-x)^2 - 4(-x^3)y = 3$$

which simplifies to

$$5x^2 + 4x^3y = 3.$$

This is not equivalent to the original equation because of the difference in one sign and not the others. Hence the graph is not symmetric about the y-axis.

Symmetry about the x-axis

If a graph is symmetric about the x-axis then for each point (x, y) on one branch there is a corresponding point $(x, -y)$ on the other branch. Such a graph is not the graph of a function (unless y is always 0), as we see since the line through any point of the graph and its reflection in the x-axis is a vertical line intersecting the graph twice.

Definition: The graph of an equation is *symmetric with respect to the x-axis* if replacing y by $-y$ in the equation gives an equivalent equation.

Example 19: To test the equation

$$3x^2y + 2x^4y^3 - 8 = 0$$

for symmetry with respect to the x-axis, replace y by $-y$ every place it appears. This gives

$$3x^2(-y) + 2x^4(-y)^3 - 8 = 0,$$

which can be simplified to

$$-3x^2y - 2x^4y^3 - 8 = 0,$$

which is not equivalent to the original equation, because two of the signs changed and one did not. This tells us that the graph of the equation is not symmetric about the x-axis.

Example 20: To test the equation

$$x^2y^3 - 6x^3y = 0$$

for symmetry with respect to the x-axis, replace y by $-y$ every place it appears. This gives

$$x^2(-y)^3 - 6x^3(-y) = 0,$$

which simplifies to

$$-x^2y^3 + 6x^3y = 0.$$

This equation is equivalent to the original equation: if we multiply both sides of this equation by -1 we get the original equation. Hence the graph is symmetric about the x-axis.

Symmetry about the Origin

Another type of symmetry uses a point as a mirror. This may seen a little odd, but is also useful. A graph is *symmetric about the origin* if for every point (x, y) on the graph, a line passing through this point and the origin intersects another point of the graph on the other side of the origin, and the same distance from the origin as the original point.

Definition: The graph of an equation is *symmetric with respect to the origin* if replacing x by $-x$ and y by $-y$ in the equation gives an equivalent equation.

Example 21: To test the equation

$$3x^2y + 2x^4y^3 - 8 = 0$$

for symmetry about the origin, replace x by $-x$ and$y$ by $-y$ every place they appear. This gives

$$3(-x)^2(-y) + 2(-x)^4(-y)^3 - 8 = 0,$$

which can be simplified to

$$-3x^2y - 2x^4y^3 - 8 = 0,$$

which is not equivalent to the original equation, because two of the signs changed and one did not. This tells us that the graph of the equation is not symmetric about the origin.

Example 22: To test the equation

$$xy^3 - 6x^3y = 0$$

for symmetry about the origin, replace $x$by $-x$ and y by $-y$ every place they appear. This gives

$$(-x)(-y)^3 - 6(-x)^3(-y) = 0,$$

which simplifies to

$$xy^3 + 6x^3y = 0.$$

This equation is equivalent to the original equation: if we multiply both sides of this equation by -1 we get the original equation. Hence the graph is symmetric about the origin.