MATH 161 — Precalculus[1]
Community College of Philadelphia

Joanne Darken
Martin Ligare

[1]Materials produced with the support of the National Science Foundation through a grant to the Middle Atlantic Consortium for Mathematics and its Applications throughout the Curriculum (MACMATC).

Math 161 — Chapter 4
Class Exercises

1. Consider the following cubic function:

$$f(x) = x^3 - 5x$$

   (a) Give the values of a_3, a_2, a_1, and a_0 for this function.

   (b) Graph the function by hand on a piece of graph paper. Label important features on your graph such as intercepts and extremal points.

   (c) Use a computer or graphing calculator to graph the function $f(x)$. Try different windows to get the best overall view of the graph.

2. Determine which of the following are cubic functions. For those that are, give the value of the leading coefficient.

   (a) $a(x) = 4x^3 + 6x^2 - 5x - 9$

   (b) $b(x) = -15x - 2 + 7x^3 + 9.6x^2$

   (c) $c(x) = x(x^2 - 1)$

   (d) $d(x) = (x - 1)(x + 3)(2x + 5)(x - 9)$

   (e) $e(x) = |10x^3 - 83x^2 - 35x - 24|$

   (f) $f(x) = 2^4 x^3 - \sqrt{4}x + \pi$

   (g) $g(x) = -1000x^3$

   (h) $h(x) = 7x^3 - 9.6x^2 - \frac{15x-2}{x}$

   (i) $i(x) = (x + 3)(2x^2 - 5x + 2)$

   (j) $j(x) = -71x^3 - 82x^5 - 17x + 2$

   (k) $k(x) = \sqrt{7x^3 - 9.6x^2 - 15x - 2}$

   (l) $l(x) = 4.3x^3 - 10$

3. Use a computer or graphing calculator to graph the functions in Problem 2.

4. Examine the graphs of all the cubic functions you have made.

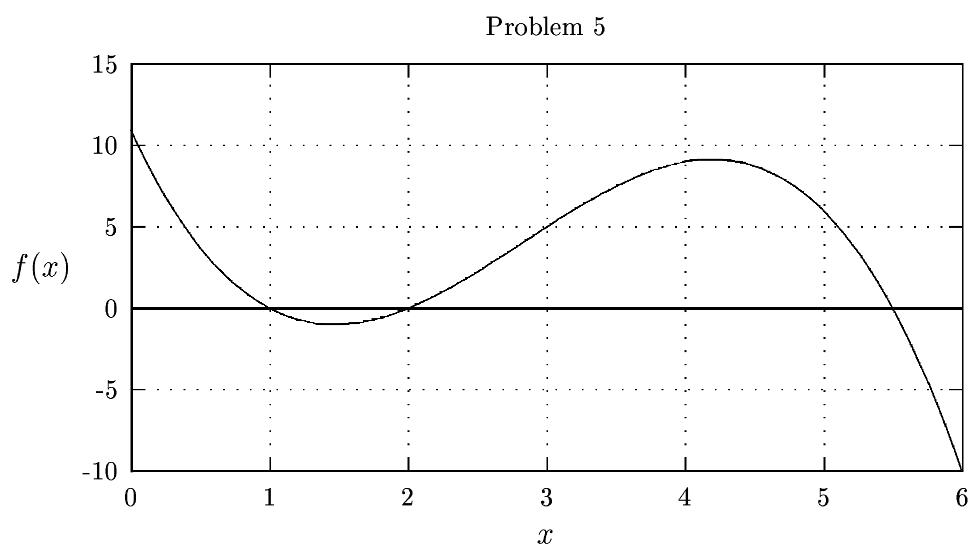
(a) What can you say about the number of y-intercepts for the graph of a cubic function?

(b) What can you say about the number of x-intercepts for the graph of a cubic function?

(c) What can you say about the number of extremal points, *i.e.*, the number of local maxima and minima, for the graph of a cubic function?

(d) What (if anything) does the sign of the leading coefficient of a cubic function tell you about the graph of the function?

5. Illustrated below is the graph of a cubic function $f(x)$.



From the graph,

(a) list approximate values of all roots,

(b) list the approximate coordinates of all local maximum and minimum points,

(c) give the limit of $f(x)$ as $x \to +\infty$; *i.e.* as x goes far to the right (assuming that all trends displayed in the graph continue outside the displayed range), and

(d) give the limit of $f(x)$ as $x \to -\infty$; *i.e.* as x goes far to the left (assuming that all trends displayed in the graph continue outside the displayed range),

(e) Give all intervals on which $f(x)$ is increasing,

(f) Give all intervals on which $f(x)$ is increasing.

6. Graph each of the following cubic functions using a computer or graphing calculator. For each of the functions

   - give approximate values for all roots of the function,
   - give approximate coordinates of all of local extrema *i.e.*, local maxima and minima, (if it has any),
   - give the limit of the function as $x \to +\infty$; *i.e.* as x goes far to the right (assuming that all trends displayed in the graph continue outside the displayed range), and
   - give the limit of the function as $x \to -\infty$; *i.e.* as x goes far to the left (assuming that all trends displayed in the graph continue outside the displayed range).

   You are going to have to play around with the displayed window to get a good picture of the function's behavior. You should display x over a large interval, *e.g.*, $[-100, 100]$, for one graph to see what happens when x gets far from zero. But you should also look at graphs with smaller displayed x intervals (intervals of length less than 10, maybe) because there are some interesting features that may not show up on graphs with large displayed domains.

   (a) $f(x) = x^3 - 3x^2 + 2x - 4$

   (b) $g(x) = 2x^3 - 11x^2 - 23x + 14$

   (c) $h(x) = x^3 - 2x^2 - 19x + 20$

   (d) $k(x) = -x^3 - 2x^2 + 3x$

   (e) $m(x) = x^3 + 3$

7. Give the formula for a cubic function answering each description given below, where possible. If not possible, explain why not.

(a) the function has roots 1, 2 and 3.

(b) the function has roots -7, 3 and 9.

(c) the function has roots -1/3, 1/2 and 10.

(d) the function goes through the origin.

(e) the function has y-intercept -6 and its only root is 1.

(f) the function has exactly 2 roots.

(g) the function has no roots.

8. Consider the function $g(x) = x^3 - 1$.

   (a) What is the domain of g?

   (b) What is the range of g?

9. Match each function given below by formula with its graph. You should be able to do this *without* making your own graph of the function.

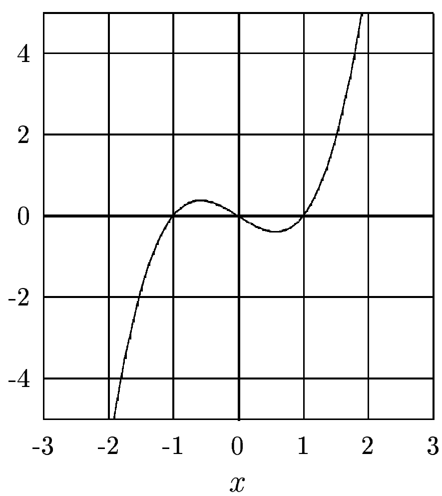
   (a) $a(x) = -x^3 + 5$

   (b) $b(x) = x^3 - x$

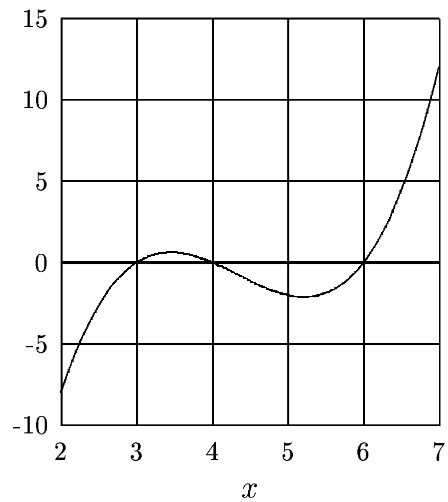
   (c) $c(x) = (x-3)(x-4)(x-1)$

   (d) $d(x) = x^3 - 13x^2 + 54x - 72$

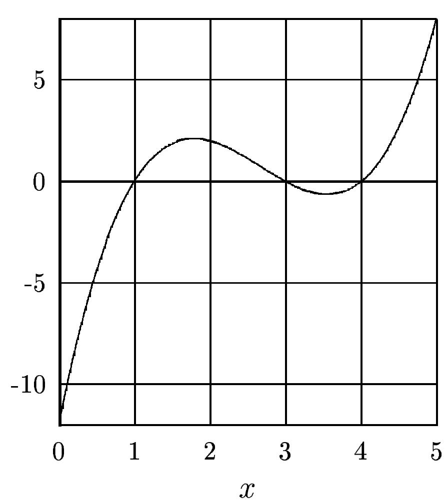
Graph #1



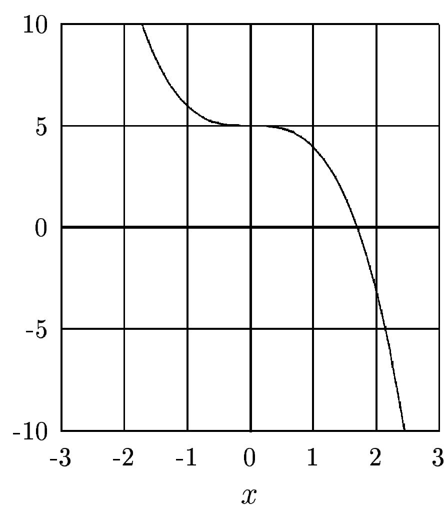
Graph #2



Graph #3



Graph #4



10. For each function in the preceding exercise, give the intervals on which it is increasing and the intervals on which it is decreasing. (Approximate where necessary.)

11. Make sketches of the graphs of cubic functions that have the following properties:

    (a) Three roots, at $x = -2$, $x = 0$, $x = 4$, and a positive leading coefficient.

    (b) One root at $x = 10$, and a local maximum at $(5, 20)$.

    (c) One root at $x = -5$, a y-intercept of -10, and a local minimum at $(-15, 5)$.

12. Let $f(x) = x^3 - 6x^2$. Find the average rate of change of f over each of the given intervals:

    (a) $[-6, -4]$
    (b) $[-4, -2]$
    (c) $[-2, 0]$
    (d) $[0, 2]$
    (e) $[2, 4]$
    (f) $[4, 6]$
    (g) $[6, 8]$

13. For the function f given in Exercise 12, graph the average rate of change over each interval versus the midpoint of the interval. (You have done similar exercises for quadratic functions in Chapter 3.)

14. For each of the following functions indicate whether or not the function is a polynomial. If it is a polynomial give the degree of the polynomial and the value of its leading coefficient.

    (a) $f(x) = 8x^3 + 4x^2 + 3x - 9$
    (b) $f(x) = 5x^2 - 8x - 2$
    (c) $f(x) = 3x - 1$

(d) $f(x) = 15x^3 + 8x^4 + 9x^2 + 7x - 8$

(e) $f(x) = 7x^5 + 8x^4 - 9x^3 + 6x^2 + 7\sqrt{x} - 1$

(f) $f(x) = \sqrt{8}x^4 - 7x - 8 + 10x^4$

(g) $f(x) = 4(x+3)^2$

(h) $f(x) = (107/x^{56}) + 8x^{47} - 5x^{31} - x^2 + (17/x) - 1$

(i) $f(x) = 67x^3 - 4x^5 + 30x - 9xx$

(j) $f(x) = -84.76 + 19.54x^2 - 35.08x$

(k) $f(x) = |9x^4 + 101x^3 - 9x^7 + 43x^2 - 8x + 502|$

(l) $f(x) = 2.1x^5 + 9.7x^4 - 9.8x^3 + 6.5x^2 + 7.0\sqrt{x} - 1.3$

(m) $f(x) = 84x^3 - 7xxxxx - 81x$

(n) $f(x) = -\pi^7x^6 - 6\pi x^4 - 2\pi^2x^3 + 49x^2 + \sqrt{\pi}x - \sqrt{6}$

(o) $f(x) = 107\sqrt{x^{56} + 8x^{47} - 5x^{31} - x^2 + (17/x) - 1}$

(p) $f(x) = (x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$

(q) $f(x) = x^3(x-3)^2(x+5)^4$

15. For each function in the exercise above that is a polynomial, give the maximum and minimum number of roots it might have, as far as you can tell by just looking at its formula, without doing any computation or graphing.

Polynomials of higher degree can be a chore to graph, even using technology, since a good overall picture often involves such large numbers that fine details don't show up, and it may be hard to find local extrema from such a graph. However, there are distinct patterns to be observed in the graphs of polynomials, and it is worth noticing these.

For the following exercises work in groups. Each group will be assigned to do **one** of the following Exercises (16- 19), and you may divide the labor as you see fit. You will be looking for the general characteristics of polynomials of a given degree. Use a computer or a graphing calculator to make graphs of the suggested functions. Then make up some polynomial functions of the same degree as those in the suggested list, and plot these also. You should be looking at overall appearance, possible numbers of roots, possible numbers of local extrema, and limiting behavior as $x \to \pm\infty$; *i.e.* gets far from zero to the left or right. You will need large windows to see some of these properties and small windows for others.

It is not valid to draw conclusions about a whole class of functions from just a few examples, but in fact the examples we have chosen do show typical polynomials of the degrees given.

For the functions you make up, it will be easier to see the properties if you use small coefficients for the higher degree terms and larger coefficients as the degrees of the terms decrease. This generally leads to more manageable and interesting graphs. For these functions you should start with a large domain, like $[-100, 100]$, to see the "global" behavior of the function. Then you can narrow down the domain. It is always possible to get a function for which you need a larger domain to see the overall behavior. (You should settle for reasonable certainty about the behavior for large and small x; as you get a better idea of what polynomial functions can possibly look like, this task will get easier.)

16. Describe (verbally and with sketches) the characteristics of the the graph of a fourth degree polynomial. In particular, describe how many local extrema it may have, how many roots, what happens as $x \to \pm\infty$, and what effect the sign of the leading coefficient has on the appearance of the graph. Graph the following four fourth-degree polynomials first. We have suggested domains; you may want to try others.

    (a) $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$ on $[-2, 5]$.

    (b) $f(x) = x^4 + 6x^3 - 7x^2 - 36x + 36$ on $[-8, 3]$.

    (c) $f(x) = -2x^4 - 21x^3 - 65x^2 - 42x + 40$ on $[-7, 3]$.

    (d) $f(x) = -x^4 + 20x^3 - 92x^2 - 272x + 1920$ on $[-5, 12]$.

17. Describe (verbally and with sketches) the characteristics of the the graph of a fifth degree polynomial. In particular, describe how many local extrema it may have, how many roots, what happens as $x \to \pm\infty$, and what effect the sign of the leading coefficient has on the appearance of the graph. Graph the following four fifth-degree polynomials first. We have suggested domains; you may want to try others.

    (a) $f(x) = x^5 - x^4 - 27x^3 + 41x^2 + 106x - 120$ on $[-6, 5]$.

(b) $f(x) = x^5 - 15x^4 + 49x^3 + 111x^2 - 482x + 336$ on $[-4, 10]$.

(c) $f(x) = -x^5 + 3x^4 + 18x^3$ on $[-5, 7]$.

(d) $f(x) = 6x^5 + 43x^4 - 249x^3 - 1581x^2 + 1435x - 294$ on $[-9, 8]$.

18. Describe (verbally and with sketches) the characteristics of the the graph of a sixth degree polynomial. In particular, describe how many local extrema it may have, how many roots, what happens as $x \to \pm\infty$, and what effect the sign of the leading coefficient has on the appearance of the graph. Graph the following four sixth degree polynomials first. We have suggested domains; you may want to try others.

    (a) $f(x) = 12x^6 + 80x^5 - 541x^4 - 2913x^3 + 4451x^2 - 2023x + 294$ on $[-9, 7]$.

    (b) $f(x) = 2x^6 - 11x^5 - 122x^4 + 134x^3 + 70x^2 - 283x - 210$ on $[-4, 6]$.

    (c) $f(x) = -x^6 - 12x^5 + 22x^4 + 560x^3 + 147x^2 - 6860x$ on $[-9, 6]$.

    (d) $f(x) = 6x^6 + 43x^5 - 249x^4 - 1581x^3 - 1435x^2 - 294x$ on $[-9, 8]$.

19. Describe (verbally and with sketches) the characteristics of the the graph of a seventh degree polynomial. In particular, describe how many local extrema it may have, how many roots, what happens as $x \to \pm\infty$, and what effect the sign of the leading coefficient has on the appearance of the graph. Graph the following four seventh degree polynomials first. We have suggested domains; you may want to try others.

    (a) $f(x) = x^7 + 12x^6 - 22x^5 - 560x^4 - 147x^3 + 6860x^2$ on $[-9, 6]$.

    (b) $f(x) = x^7 + 5x^6 + 3x^5 - 17x^4 - 16x^3 + 24x^2 + 16x - 16$ on $[-3, 2]$.

    (c) $f(x) = -27x^7 - 270x^6 + 531x^5 + 4796x^4 - 17205x^3 + 20050x^2 - 7875x$ on $[-9.4, 5]$.

    (d) $f(x) = x^7 - 10x^6 + 33x^5 - 36x^4$ on $[-1, 4.3]$.

20. After hearing the results of other groups, describe the characteristics you expect in the graph of a polynomial of degree n, where n is a whole number. How many roots might it have? How many local extrema? What is the behavior as $x \to \pm\infty$? What does the leading coefficient tell you?

21. Make a quick sketch of the graph of a polynomial that fits each of the following descriptions. Some may be impossible – say so if that's the case.

    (a) 4^{th} degree, positive leading coefficient, 4 roots
    (b) 3^{rd} degree, negative leading coefficient, 3 roots
    (c) 4^{th} degree, positive leading coefficient, no roots
    (d) 5^{th} degree, positive leading coefficient, 3 roots
    (e) 2^{nd} degree, negative leading coefficient, 3 roots
    (f) 7^{th} degree, positive leading coefficient, 4 roots
    (g) 9^{th} degree, negative leading coefficient, no roots

22. For each of the following cubic functions, make a table with a column for each term and one for the function itself, and a row for each of: $x = 5$, $x = 10$, $x = 20$, $x = 50$, $x = 100$. Complete the table with the appropriate values.

    (a) $P_1(x) = x^3 + x^2 + 3x + 1$
    (b) $P_2(x) = x^3 - x^2 + 3x + 1$
    (c) $P_3(x) = -x^3 + x^2 - 7x - 30$
    (d) $P_4(x) = -x^3 - x^2 - 7x - 30$

23. For the following refer to the cubic functions in the preceding exercise and the values of the outputs obtained. (You are asked how much one function value differs from another in percentage terms. To find this, divide the first value by the second. If the answer is, for example, 1.43, then the first function is 43 percent bigger than the second for that value of x.)

    (a) Find how much $P_2(x)$ differs from $P_1(x)$ in percentage terms for each value of x.
    (b) Find a value of x for which $P_2(x)$ differs from $P_1(x)$ by less than one percent.
    (c) Graph both functions in a window in which the two graphs cannot be distinguished.

(d) Name another graph that cannot be distinguished from these two in the window you chose.

(e) Find how much $P_4(x)$ differs from $P_3(x)$ in percentage terms for each value of x.

(f) Find a value of x for which $P_4(x)$ differs from $P_3(x)$ by less than one percent.

(g) Graph both functions in a window in which the two graphs cannot be distinguished.

(h) Name another graph that cannot be distinguished from these two in the window you chose.

24. For each of the following functions, make a table giving the value of the function and the value of its leading term for: $x = 5$, $x = 10$, $x = 20$, $x = 50$, $x = 100$

(a) $P_1(x) = x^7 + 2x^5 + 3x^4 + -2x^2 - 5$

(b) $P_2(x) = x^6 - x^3 + 3x + 1$

(c) $P_3(x) = -x^5 + 3x^2 - 7x - 30$

(d) $P_4(x) = -x^4 - x^3 - 7x - 30$

25. For each function in the preceding exercise, graph the function itself and the function given by its leading term in a window in which the two graphs cannot be distinguished.

26. Evaluate the following expressions:

(a) $lim_{x\to+\infty}(8x^3 + 4x^2 + 3x - 9)$

(b) $lim_{x\to-\infty}(5x^2 - 8x - 2)$

(c) $lim_{x\to+\infty}(-8x^4 + 15x^3 + 9x^2)$

(d) $lim_{x\to-\infty}(7x^5 + 8x^4 - 9)$

(e) $lim_{x\to-\infty}(-8 - 7x + 8x^4)$

(f) $lim_{x\to+\infty}(7x - 9)$

(g) $lim_{x\to-\infty}(0.00005x^2 - 80000x - 3)$

(h) $lim_{x\to+\infty}(-x^6 + 15x^5 + 4x^2)$

(i) $lim_{x \to -\infty}(-x^{15} + 8x^10 - 9x)$

(j) $lim_{x \to -\infty}(-2 - 70x + 800x^4)$

(k) $lim_{x \to -\infty}(8x^3 + 4x^2 + 3x - 9)$

(l) $lim_{x \to +\infty}(5x^2 - 8x - 2)$

(m) $lim_{x \to -\infty}(-8x^4 + 15x^3 + 9x^2)$

(n) $lim_{x \to +\infty}(7x^5 + 8x^4 - 9)$

(o) $lim_{x \to +\infty}(-8 - 7x + 8x^4)$

(p) $lim_{x \to -\infty}(7x - 9)$

(q) $lim_{x \to +\infty}(0.00005x^2 - 80000x - 3)$

(r) $lim_{x \to -\infty}(-x^6 + 15x^5 + 4x^2)$

(s) $lim_{x \to +\infty}(-x^{15} + 8x^10 - 9x)$

(t) $lim_{x \to +\infty}(-2 - 70x + 800x^4)$

27. Answer the following questions:

    (a) How many roots does the polynomial $P(x) = (x-1)(x-2)(x-3)$ have?

    (b) How many local extrema does the polynomial $P(x)$ from part 27a have?

    (c) How many roots does the polynomial $F(x) = x^5 + 7$ have?

    (d) How many local extrema does the polynomial $F(x)$ from part 27c have?

28. Make a graph of the function $f(x) = 2(x-3)^2$.

    (a) What are the coordinates of the vertex of the parabola?

    (b) Invent a new function that has exactly the same shape as f but has a vertex that is 4 units directly above the vertex of f. Check your results by making a graph of this new function.

    (c) Invent a new function that has exactly the same shape as f but has a vertex that is 5 units directly below the vertex of f. Check your results by making a graph of this new function.

29. Make a graph of the function $h(x) = -.1x^3 + 3x$.

(a) What are the coordinates of the local extreme points of your graph?

(b) What are the roots of the function h?

(c) Invent a new function k that has exactly the same shape as h but has every point shifted straight up by 4 units. Check your results by making a graph of this new function.

(d) Invent a new function m that has exactly the same shape as h, but has every point shifted straight down by 2 units.

(e) What are the approximate roots of your new functions k and m.? Are they the same as before?

(f) Invent a new function that has exactly the same shape as h, but that has only 1 root.

30. Consider the function $p(x) = x^2 + 2$.

(a) Invent a new function that has exactly the same shape as p but has a vertex that is 1 unit to the right of the vertex of p (along a horizontal line). Check your results by making a graph of this new function.

(b) Invent a new function that has exactly the same shape as p but has a vertex that is 3 units to the left of the vertex of p (along a horizontal line). Check your results by making a graph of this new function.

31. Once again consider the function $h(x) = -.1x^3 + 3x$.

(a) Invent a new function that has exactly the same shape as h but has every point shifted 2 units to the right (along a horizontal line). Check your results by making a graph of this new function.

(b) What are the approximate coordinates of the local extrema of your new function?

(c) What are the approximate roots of your new function?

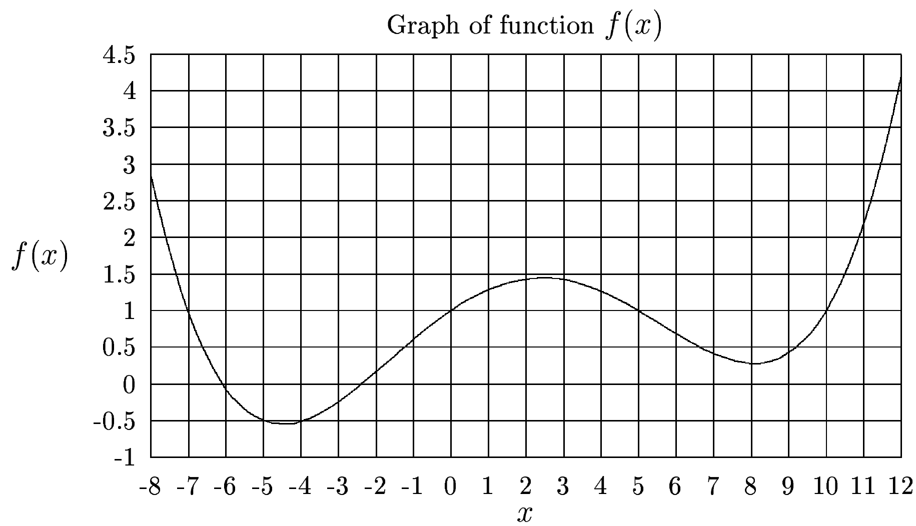
32. Consider the function $r(x) = x^3 - 9x^2 - 9x + 81$.

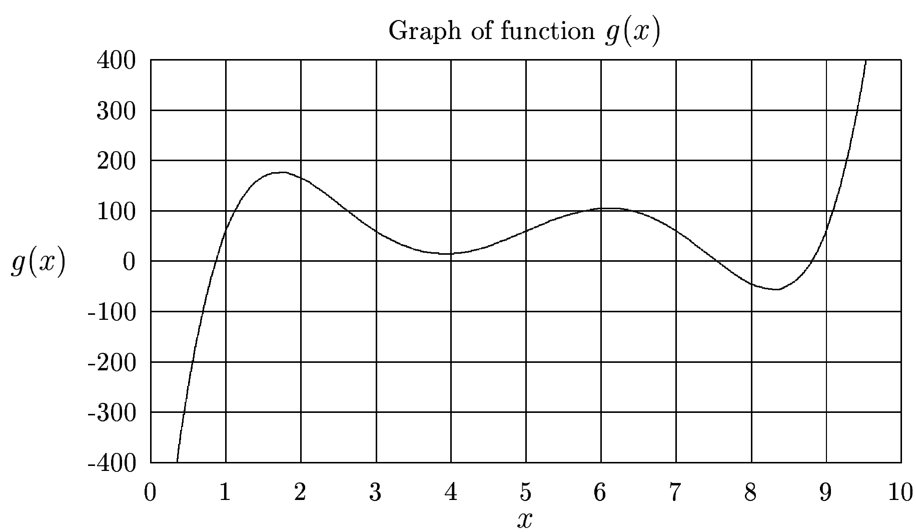
(a) Make a graph of the function $r(x)$.

(b) Invent a new function that has exactly the same shape as r but has every point shifted 1 unit to the left (along a horizontal line) **and** shifted down 2 units vertically.

(c) Invent a new function that has exactly the same shape as r but has only 1 root.

33. Below are the graphs of two polynomial functions $f(x)$ and $g(x)$. Assume these graphs give a fair picture of the functions they represent.





Another polynomial function is given algebraically by

$$n(x) = -6x^3 + 2x^2 - 189x + 64.$$

(a) What are the roots of $f(x)$?

(b) What are the coordinates of the local maxima of $f(x)$?

(c) What are the coordinates of the local minima of $g(x)$?

(d) What are the intervals of increase of $f(x)$?

(e) What are the intervals of decrease of $f(x)$?

(f) What are the intervals of increase of $g(x)$?

(g) What are the intervals of decrease of $g(x)$?

(h) What is the sign of the leading coefficient of $g(x)$.

(i) What is the limit of $f(x)$ as $x \to +\infty$? (Assume that all local extrema are visible on the graph.)

(j) What is the limit of $n(x)$ as $x \to -\infty$?

(k) What is the y-intercept of $f(x)$?

(l) What is the y-intercept of $n(x)$?

(m) What is the degree of $n(x)$?

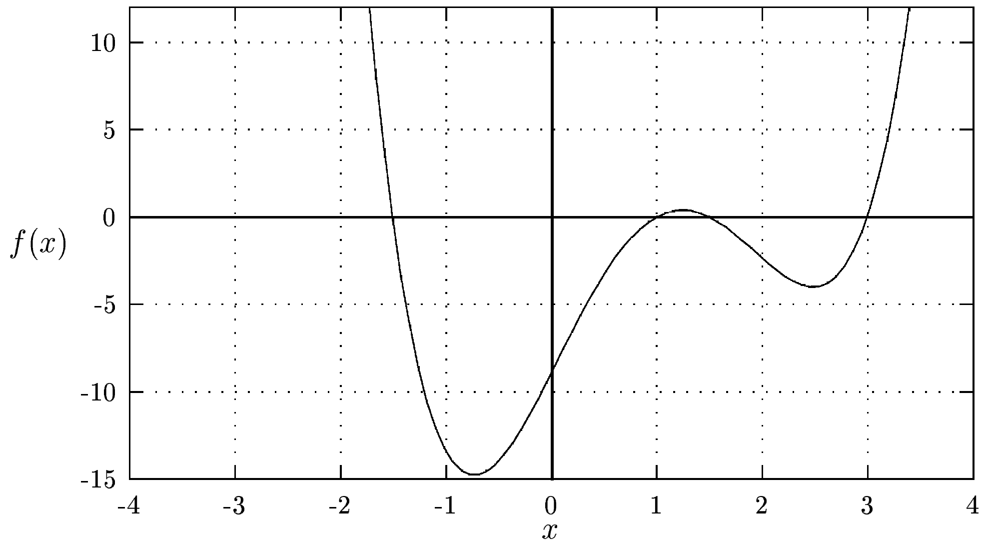
(n) Is the degree of $f(x)$ odd or even? (Assume that all local extrema are visible in the graph.)

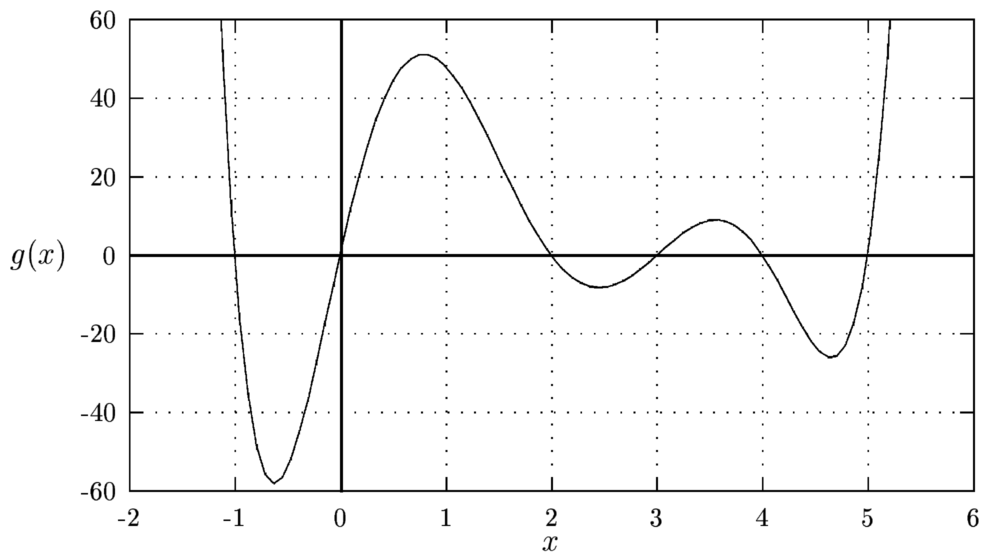
(o) What is the minimum possible degree of $f(x)$?

(p) What is the average rate of change of $g(x)$ over the interval $[5, 7]$?

(q) What is the average rate of change of $n(x)$ over the interval $[0, 2]$?

34. Below are the graphs of two polynomial functions $f(x)$ and $g(x)$.





Another polynomial function is given algebraically by

$$n(x) = 2x^3 + 6x^2 - 12x - 16$$

(a) What are the roots of $f(x)$?

(b) What are the coordinates of the local maxima of $f(x)$?

(c) What are the coordinates of the local minima of $g(x)$?

(d) What are the intervals of increase of $f(x)$?

(e) What are the intervals of decrease of $f(x)$?

(f) What are the intervals of increase of $g(x)$?

(g) What are the intervals of decrease of $g(x)$?

(h) What is the sign of the leading coefficient of $g(x)$.

(i) What is the limit of $f(x)$ as $x \to +\infty$? (Assume that all local extrema are visible on the graph.)

(j) What is the limit of $n(x)$ as $x \to -\infty$?

(k) What is the y-intercept of $f(x)$?

(l) What is the y-intercept of $n(x)$?

(m) What is the degree of $n(x)$?

(n) Is the degree of $f(x)$ odd or even? (Assume that all local extrema are visible in the graph.)

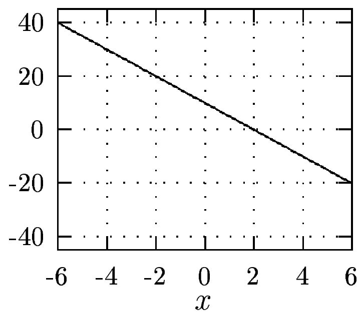
(o) What is the minimum degree of $f(x)$?

(p) What is the average rate of change of $g(x)$ over the interval $[5, 7]$?

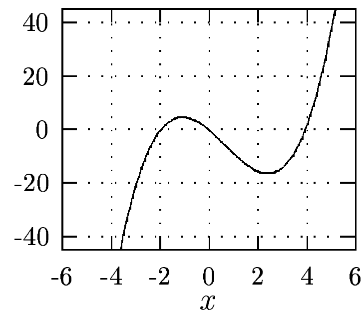
(q) What is the average rate of change of $n(x)$ over the interval $[0, 2]$?

35. For each of the illustrated graphs of polynomials on the following page, imagine calculating the rate of change over all of the unit intervals $[0, 1]$, $[1, 2]$, etc., and plotting this rate of change versus the midpoint of the interval. We have done this for you, and all we want you to do is match the graphs with the correct rate of change graphs. **NOTE**: You don't have to actually work out the actual values of the rates of change to do this problem.

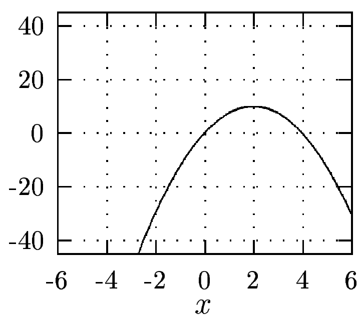
$A(x)$



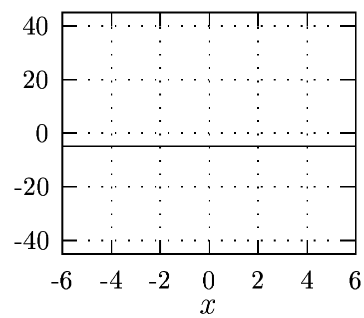
aroc #1



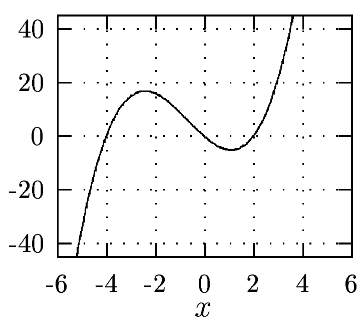
$B(x)$



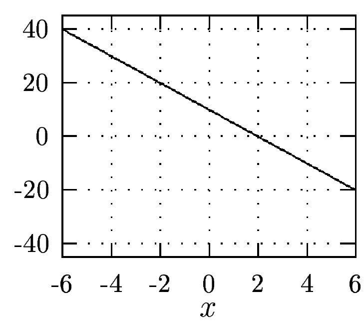
aroc #2



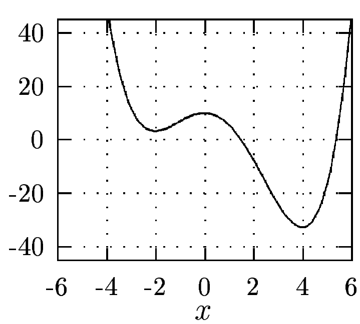
$C(x)$



aroc #3



$D(x)$



aroc #4