MATH 161 — Precalculus[1]
Community College of Philadelphia

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[1]Materials produced with the support of the National Science Foundation through a grant to the Middle Atlantic Consortium for Mathematics and its Applications throughout the Curriculum (MACMATC).

Math 161 — Chapter 4
Polynomials
Information

4.1 Introduction

We now study a class of functions, polynomials, that includes linear and quadratic functions, as well as functions in which the variable is raised to higher powers. Higher-degree polynomials are worth our attention because they are needed in calculus as tools for approximating other functions. Also, they provide excellent examples of certain important function properties. Rates of change of polynomials have some interesting properties which we will also consider.

4.2 Power functions

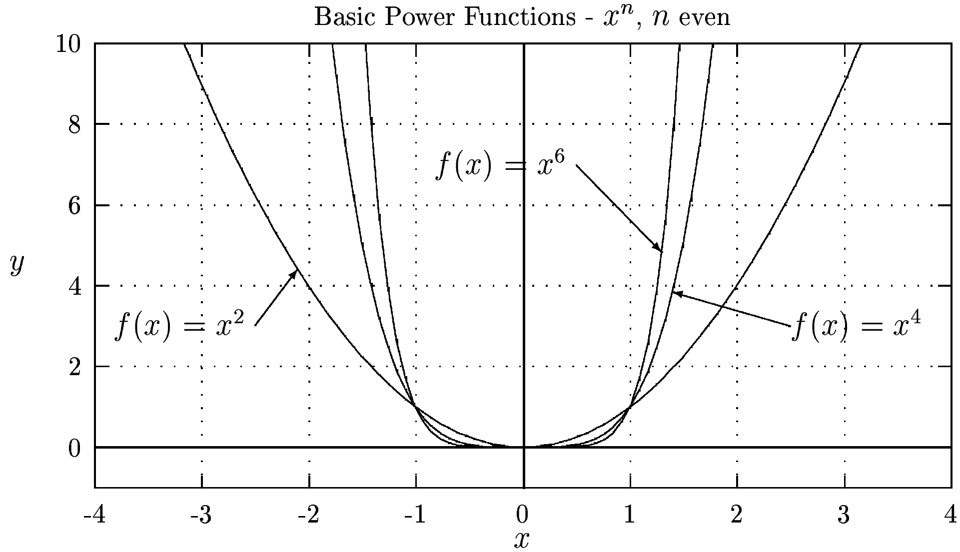
A *power function* is a function of the form $f(x) = ax^n$. The exponent may be any whole number: 0, 1, 2, 3, ... (The term *power function* is used in some books to describe any function of this form using any real number exponent, but we do not do that.) The coefficient may be any real number. The *degree* of a power function $f(x) = ax^n$ with $a \neq 0$ is n.

Example 1: The linear function $f(x) = x$ is a power function of degree 1.

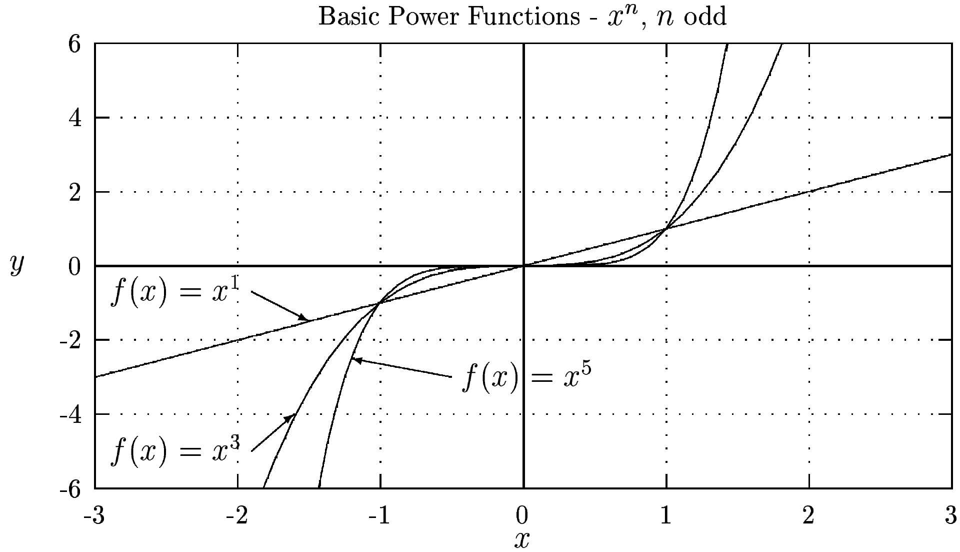
Example 2: The quadratic function $f(x) = x^2$ is a power function of degree 2.

Example 3: Each of the following is a power function: $f_1(x) = x^3$, $f_2(x) = -4x^4$, $f_3(x) = \frac{2}{5}x^4$, $f_4(x) = 871x^{62}$, $f_5(x) = 3x^0 = 3$. In the last case observe that we have a constant function. We are even less likely to think of this as a power function than we are to think of the linear and quadratic examples that way; each is a special case which has its own name.

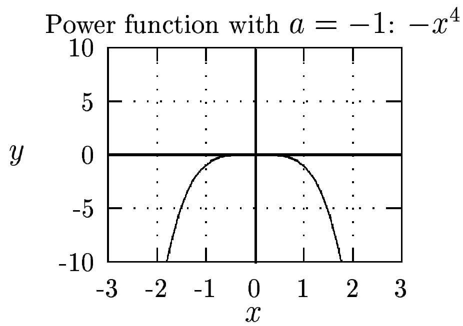
For n even (*i.e.*, 2, 4, 6, 8, *etc.*) the graph of $f(x) = x^n$ resembles a parabola in shape, though as the degree gets higher the graph gets flatter near the origin and steeper elsewhere. Simple power functions are illustrated below. The graph of $f(x) = ax^n$ will be similar, but the exact shape will depend on the value of a. If a is positive, the graph will open upward; it will open downward if a is negative.

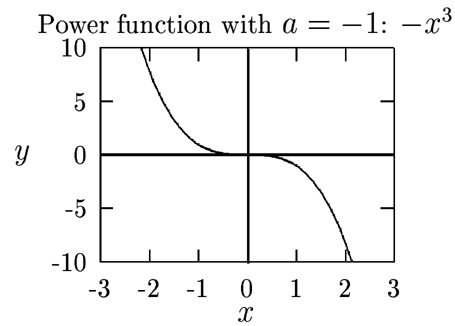


For n odd (*i.e.*, 2, 4, 6, 8, *etc.*) the graph of $f(x) = x^n$ resembles the graph shown below. As with the even-degree polynomials, the higher the degree the flatter the graph is near the origin and the steeper elsewhere.



In addition to the simplest power functions with $a = 1$ (examples of which have been illustrated above) the functions $f(x) = -x^n$ (here $a = -1$) are of particular interest. Two typical graphs are shown below.





4.3 Polynomials

Definition and examples of polynomials

Definition: A *polynomial function* is a function that can be put in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x + a_0$$

where n is a whole number (0, 1, 2, etc.) and a_n, a_{n-1}, a_{n-2}, $...a_2$, a_1, a_0 are real numbers.

The value of n is the *degree* of the polynomial, assuming $a_n \neq 0$. The degree is important; for example, polynomials of different degrees have significant differences in their graphs.

The a_n, a_{n-1}, a_{n-2}, $...a_2$, a_1, a_0 are the *coefficients* of the function, and the coefficient of the term of highest degree is the *leading coefficient.*

We have already seen some polynomial functions: Besides linear and quadratic functions, every power function is a polynomial. In fact, a polynomial could be defined as a sum of power functions.

Example 4: A linear function $f(x) = mx+b$ is a special case of a polynomial. In this case $n = 1$, $a_n = a_1 = m$, and $a_0 = b$. The degree of a linear function is 1. Note that in this case we do not see as many terms as in the definition of polynomial function, which is written to show the pattern for higher degree polynomials. If we write the linear function in the general polynomial form we have $f(x) = a_1x + a_0$. This is rarely done, and linear functions are so special that we rarely even think of them as polynomials.

Example 5: A quadratic function $f(x) = ax^2 + bx + c$ is another special case of a polynomial. In this case $n = 2$, $a_n = a_2 = a$, $a_{n-1} = a_1 = b$, and $a_0 = c$. If we wrote the quadratic using the general notation we would have $f(x) = a_2x^2 + a_1x + a_0$.

Example 6: $f(x) = x^2 + 7x - 2$.
Here $n = 2$, $a_2 = 1$, $a_1 = 7$, and $a_0 = -2$. The function f is a second degree polynomial (a quadratic function) with leading coefficient 1.

Example 7: $P(x) = 3x - 8$.
Here $n = 1$, $a_1 = 3$, and $a_0 = -8$. The function P is a first degree polynomial (a linear function) with leading coefficient 3.

Example 8: $g(x) = 5x^4 + 2x^2 + x - 4$.
Here $n = 4$, $a_4 = 5$, $a_3 = 0$, $a_2 = 2$, $a_1 = 1$, and $a_0 = -4$. The function g is a fourth degree polynomial with leading coefficient 5.

Example 9: $Q(x) = x^{52} - \pi$.
Here $n = 52$, $a_{52} = 1$, $a_{51} = a_{50} = \ldots a_2 = a_1 = 0$, and $a_0 = -\pi$. The function Q is a 52^{nd} degree polynomial with leading coefficient 1.

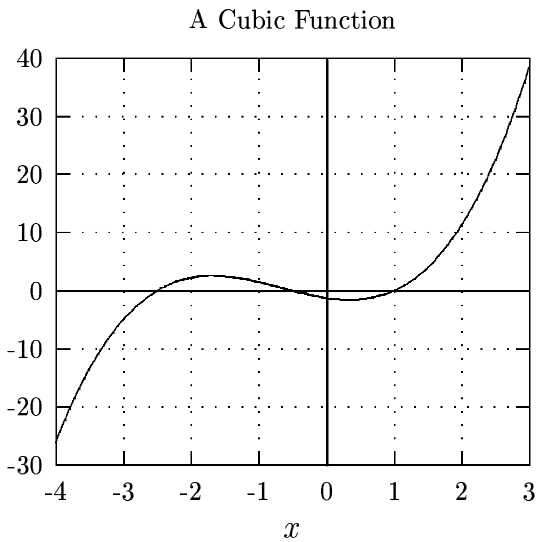
Example 10: $h(t) = 6t + (t+2)^2 = 6t + t^2 + 4t + 4 = t^2 + 10t + 4$
Here, after expanding and collecting terms, we see that h is a second degree polynomial ($n = 2$) with $a_2 = 1$, $a_1 = 10$, and $a_0 = 4$.

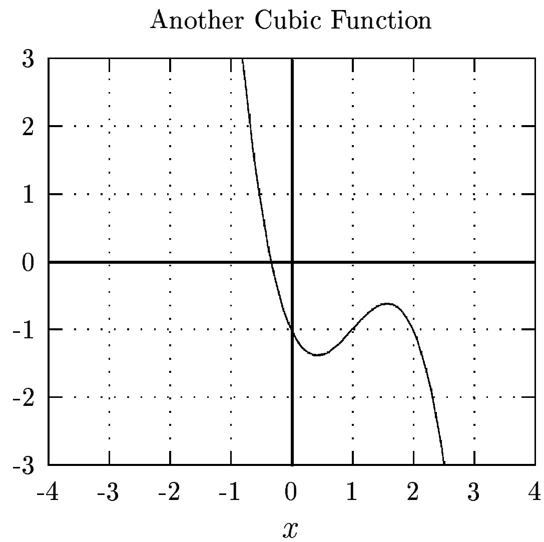
Cubics

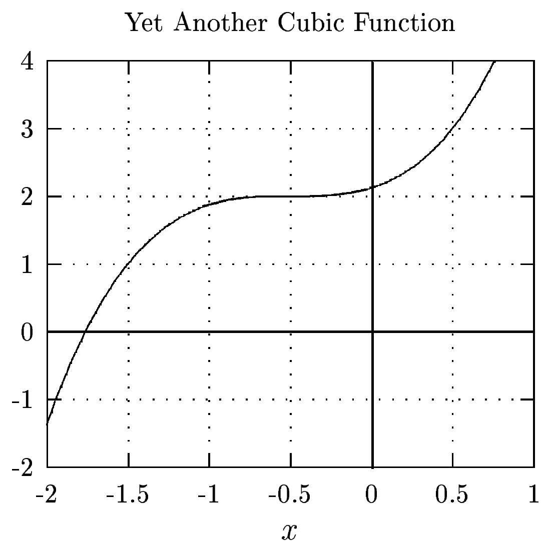
A *cubic* is a polynomial of degree 3. We now look at cubics in some detail as an introduction to analysis of higher-degree polynomials. The general form of a cubic is

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

The graphs of cubic functions have somewhat more variety than the graphs of quadratics, but still fall into certain distinctive types. Several are shown below.







Roots of polynomials

A *root* of a polynomial $P(x)$ is a number r such that $P(r) = 0$. Graphically, a root is an x-intercept; *i.e.*, the x-value of a point at which the graph of the polynomial intersects the x-axis. A root is also called a *zero* of the polynomial.

In studying quadratic functions we learned everything there is to know about their roots: if they exist they can be found by factoring or by use of the quadratic formula, and they may not exist. If not, the discriminant tells us that. For higher-degree polynomials, the good news is that if they factor, roots can still be found by factorization (using the Principle of Zero Products as before). The bad news is that if they don't factor, there's nothing like the quadratic formula to give the roots. For polynomials of degree three or four, there are formulas, but nobody uses them because they're too complicated. For polynomials of degree higher than four, there are no formulas, and it's not that they haven't been figured out yet, it's that they have been proved not to exist. For higher-degree polynomials, finding roots is usually a matter of approximating them, using a calculator or computer. There are other approximation methods, developed before the invention of calculators and computers, but we will not go into them. (You may see one in calculus.) The only non-technical method we will use is estimation from graphs.

An important algebraic point about roots is that every root of a polynomial P corresponds to a factor of P, if we allow factorization with non-integer coefficients. For example, let $P(x) = x^2 - 3$. The roots are $\pm\sqrt{3}$, and $P(x) = (x + \sqrt{3})(x - \sqrt{3})$. Since a polynomial of degree n cannot have more than n linear factors, it cannot have more than n roots.

Example 11: Let $P(x) = 4(x - 1)(x - 2)(x - 3)$. The roots of P are 1, 2 and 3, since $P(1) = P(2) = P(3) = 0$.

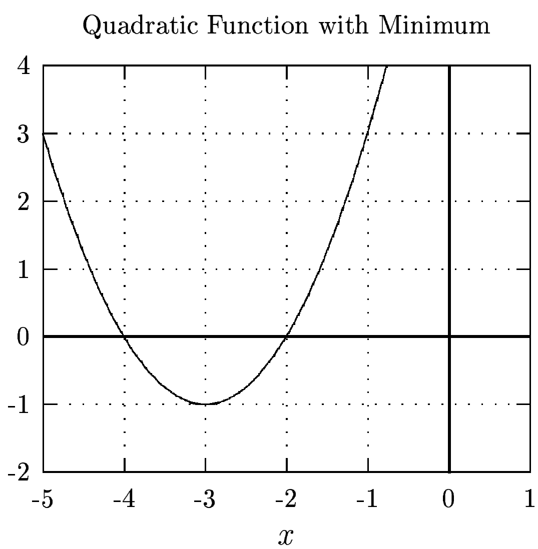
Example 12: Let $P(x) = x^2 + 1$. This quadratic has no roots. This can be seen from its graph, which doesn't touch the x-axis, from the quadratic formula (the discriminant is negative) or by observing that since $x^2 \geq 0$ for all x, $P(x)$ is always greater than or equal to 1. Or observe that the vertex of the parabola is $(0, 1)$ and the parabola opens upward.

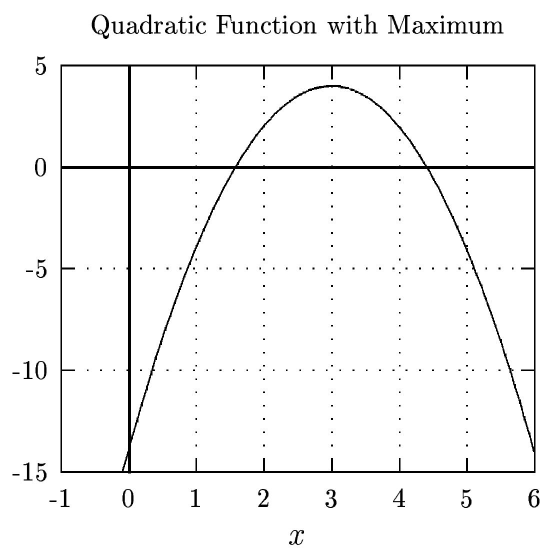
Example 13: Let $P(x) = x^3 - 4x^2 + x + 6$. It is possible to factor $P(x)$, but it may not be obvious how. If not, use a calculator or computer to solve, or make a graph and estimate, then substitute into the formula to check. The roots are 2, 3, and -1. (When estimating from a graph, always check any integer answers you seem to get by substituting them into the function formula, in case the true answer is not exactly equal to the integer.)

Example 14: Let $P(x) = x^3 - 3x^2 + x + 4$. This does not factor. You can use the technology at your disposal to solve it approximately. It has one root, approximately -0.89.

Extrema of Polynomials

The *maximum* of a function is the largest value it takes, and the *minimum* is the smallest. As observed above, a function need not have a maximum or a minimum, or it may have one but not the other. For example, if the graph of a quadratic function opens upward, it has a minimum but not a maximum. If its graph opens downward it has a maximum but not a minimum. In the illustration below the parabola on the left opens upward, and the function it represents has a minimum of -1 at $x = -3$. But it has no maximum, because the values of the function get larger and larger without bound the further we look to the left or right. The right hand parabola opens downward, and has a maximum of 4 at $x = 3$. It has no minimum, because the values of the function get lower and lower without bound the further we look to the left or right.





Any quadratic function has a minimum or maximum y-value, which it takes at its vertex. If you look at the three graphs of cubic polynomials shown in the section above on cubics, you see that none of them has a minimum or maximum y-value, at least in the window shown. In fact, no cubic has either a maximum or a minimum y-value. But the first two cubics graphed do each have a *local maximum* and a *local minimum*; that is, a y-value that is higher than any in a little patch of the graph near it, and a y-value that is lower than any in a little patch of the graph near it. In the first graph the local maximum is approximately 2 and occurs when x is approximately -1.8; the local minimum is approximately -1 and occurs when x is approximately 0.4. In the second graph the local maximum is approximately -0.6 and occurs for x approximately 1.7. Such a local maximum or minimum is called a local *extremum* (plural *local extrema.*)

We say that a function has a *local maximum* at x_0 if there is an open interval around x_0 such that the value of $f(x_0)$ is larger than $f(x)$ for any other x in the interval. So $(x_0, f(x_0)$ is a hill top). We say the function has a *local minimum* if there is an open interval around x_0 such that the value of $f(x_0)$ is smaller than $f(x)$ for any other x in the interval. (So $(x_0, f(x_0))$ is a valley bottom).

The large and small pictures

A good graph of a polynomial should show its roots and local extrema, as well as give a good idea of its overall shape.

Most of the polynomials you encounter in this course have coefficients that are integers and reasonably small. Once you know what type of graph to expect for a polynomial of a particular degree, getting a good graph is mostly a matter of experimenting with windows till you see something plausible.

However, you should be aware that when less friendly coefficients are used it is possible to create a polynomial P with roots or local extrema that don't show up in what seems to be a good graph of P, because they are beyond the boundaries of the window used or, in the case of extrema, because the bends created are too small to see. Calculus provides methods for tracking these down. For present purposes, you should be aware that sometimes you may need to use more than one graph to show all the important features.

The large picture of a polynomial P of degree n has several important features:

- The domain consists of all real numbers.
- The graph is continuous; that is, there are no breaks in it — you can draw it without lifting pencil from paper.

- When viewed from far enough away, the graph of any polynomial has the shape of the power function represented by its leading term, $a_n x^n$. All the bends are too small to see, and any and all roots are squeezed in near the origin, too close together to distinguish.
- As x gets very large in absolute value, the value of $P(x)$ does also. It is positive or negative depending on whether the sign of the leading coefficient is positive or negative.
- The number of roots is no greater than n and the number of extrema is no greater than $n-1$.

Note: Referring to the graphs of power functions, we see that the second and third points above have an implication for roots of odd-degree polynomials: since the graph of an odd-degree power function goes continuously from down low on the left to up high on the right or *vice versa*, it must cross the x-axis. So an odd-degree polynomial must have at least one root. (This need not be true for an even-degree polynomial, as we saw even with quadratics.)

4.4 Limits and limit notation

For sketching graphs and for other purposes it is important to consider the behavior of a polynomial function as x gets very large in absolute value, whether it's positive or negative, so that we have a sense of what happens beyond what we actually see. Some of the exercises provide an opportunity to see what is going on and why.

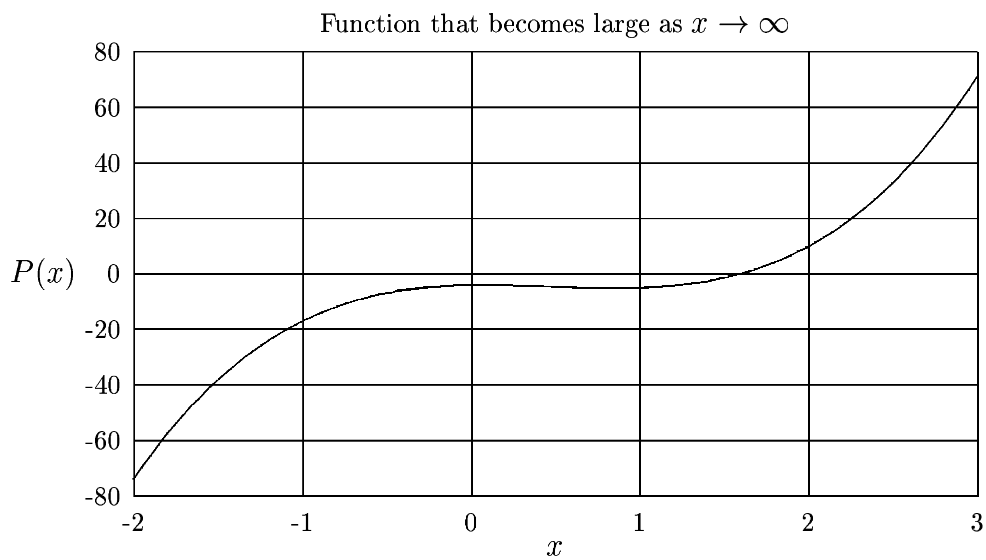
We now introduce some notation that is used in calculus to replace descriptions of graphs such as "down low on the left" and "up high on the right." The new terminology can be generalized and can also be defined precisely, although we give only intuitive definitions here The symbol ∞ that you see below stands for *infinity*, which can be interpreted as "far, far away."

1. To say that when x takes large positive values, $P(x)$ does also, we write

$$\lim_{x \to +\infty} P(x) = +\infty.$$

   This is read "As x gets large without bound, $P(x)$ gets large without bound." The graph of any function for which this is true eventually keeps rising as it goes to the right past any y-value you choose.

Example 15: Let $P(x) = 5x^3 - 7x^2 + x - 4$. A graph of this appears below. The leading coefficient of the function is positive, and as x gets large the value of $P(x)$ gets large–in fact, a lot larger than x. (In all discussion and examples, the term "large" is relative. In this example, we get a fairly good graph on the interval $[-10, 10]$, the y-values being in the thousands. But if the example had been $P(x) = 0.5x^3 - 7000x^2 + x - 4$ we would have to go from about -20000 to 20000 on the x-axis and into the billions on the y-axis to see a good picture.)

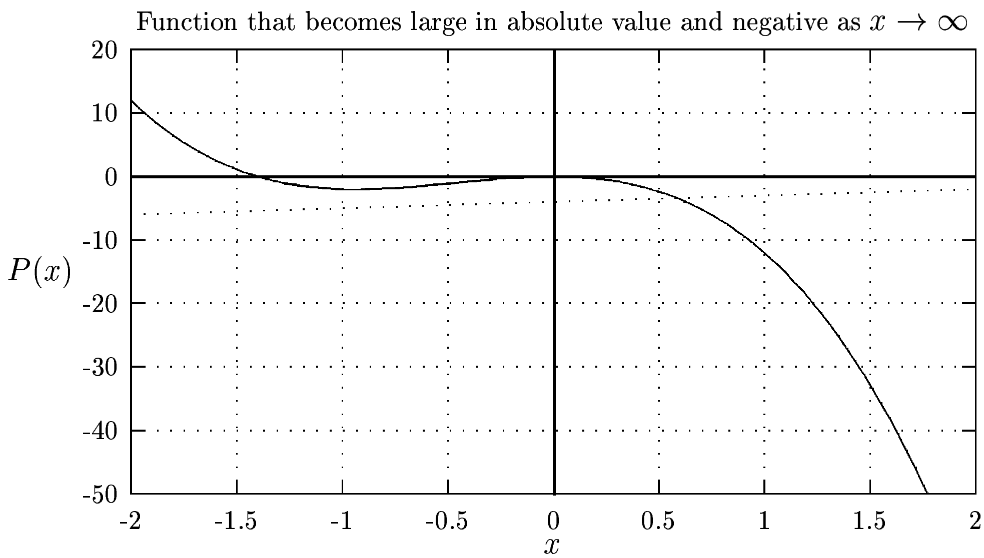


2. To say that when x takes large positive values, $P(x)$ is negative and large in absolute value, we write

$$\lim_{x \to +\infty} P(x) = -\infty.$$

This is read "As x gets large without bound, $P(x)$ becomes large without bound in absolute value and negative." The graph of any function for which this is true eventually keeps falling as it goes to the right, below any y-value you choose.

Example 16: Let $P(x) = -5x^3 - 7x^2 + x - 4$. A graph of this function appears below. The leading coefficient of the function is negative and as x gets large the value of $P(x)$ gets large in absolute value and negative. For any y value you choose, however far down, there is an x such that $P(x) = y$.



3. To say that when x is negative and far from zero $P(x)$ is large, we write

$$\lim_{x \to -\infty} P(x) = +\infty.$$

   This is read "As x gets large in absolute value and negative, $P(x)$ gets large without bound." The graph of any function for which this is true eventually keeps rising as it goes to the left.

   An example is the polynomial $P(x) = -5x^3 - 7x^2 + x - 4$ of the preceding example. For values of x that are far to the left of the origin $P(x)$ is large. Corresponding points of the graph are therefore way to the left and way up.

4. To say that when x is negative and far from zero, $P(x)$ is also, we write

$$\lim_{x \to -\infty} P(x) = -\infty.$$

This is read "For values of x that are negative and far from zero, $P(x)$ is far from zero and negative." The graph of any function for which this is true eventually keeps falling as it goes to the left.

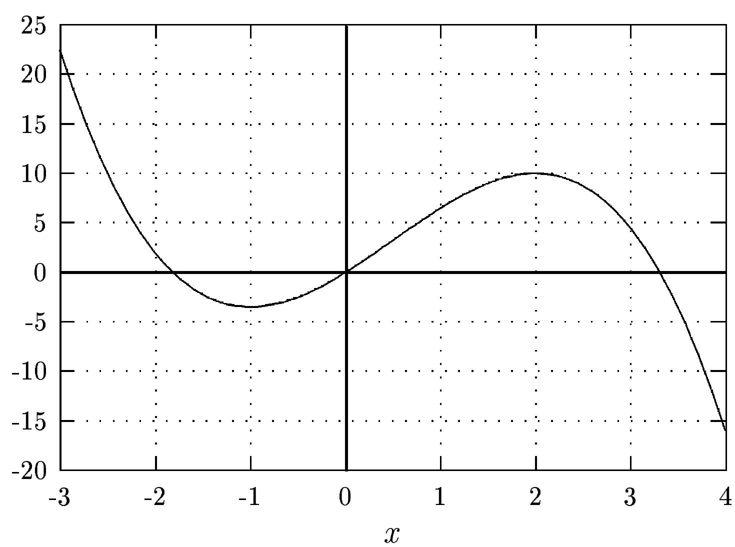
The polynomial $P(x) = 5x^3 - 7x^2 + x - 4$ used above shows this behavior.

4.5 Increasing and Decreasing Functions

A function f is said to be *increasing* on an interval I if its graph rises as it goes to the right. In other words, if x_1 and x_2 are in the interval I, and $x_1 < x_2$ then $f(x_1) < f(x_2)$.

A function f is said to be *decreasing* on an interval I if its graph falls as it goes to the right. In other words, if x_1 and x_2 are in the interval I and $x_1 < x_2$ then $f(x_1) > f(x_2)$.

In the diagram below, the function shown is decreasing on the interval $[-3, -1]$, increasing on the interval $[-1, 2]$ and decreasing on the interval $[2, 4]$.



4.6 Average Rate of Change for Polynomials

Many important questions concern rate of change: determining how fast an object is moving, how quickly a population is increasing, at what rate the blood level of a medicine is decreasing. If something changes at a steady rate, the question is usually easy to answer. But often the changes of interest are not steady. Sometimes the rate of change changes. Calculus provides tools for finding rates of change even if they are constantly changing, as long as they do so in a reasonably orderly manner. Rates of change of polynomials behave in a very orderly manner, and we will now investigate them, as a way of preparing you to deal with the concepts and methods you will study in calculus.

Recall that the rate of change of a function is defined over an interval. In the case of a linear function, the rate of change is the same over any interval; it is the slope of the line. But in the case of polynomials (and all non-linear functions) it's different on most intervals. If you find the rate of change of a function on an interval, then split the interval in two and find the rate of change on each of the two new intervals, you get two new answers. But the more you keep on doing this, the less the new answers differ from the old ones. This is the basic observation underlying concepts and techniques made precise in calculus. Here we do some work with average rates of change of polynomials and observe a pattern that exists.

What we do first (and something along these lines appears in exercises in previous chapters) is take a particular function and set of intervals, and make a table of the average rate of change of the function over these intervals.

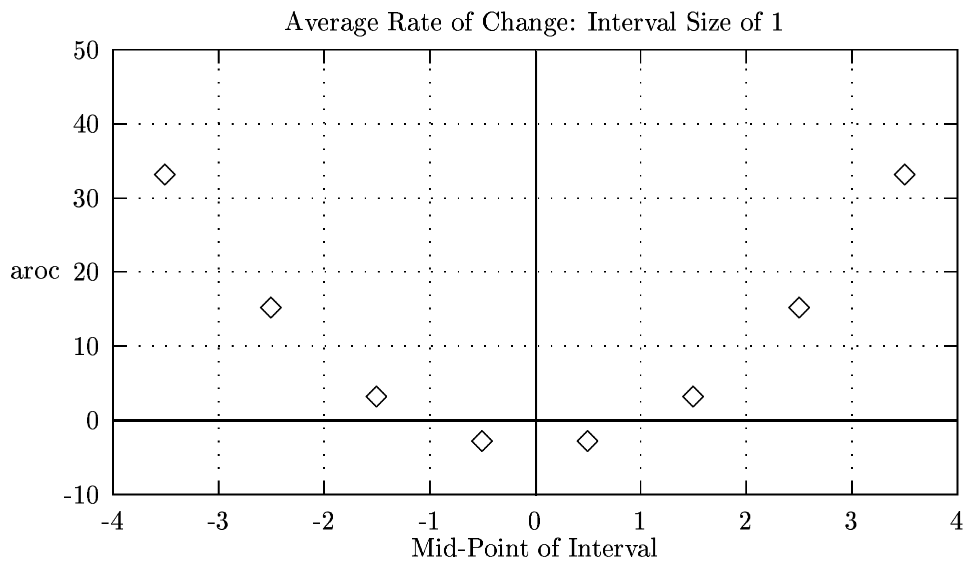
Example 17: Let $f(x) = x^3 - 4x$. Using the average rate of change formula

$$\frac{f(b) - f(a)}{b - a}$$

and the intervals $[-4, -3]$, $[-3, -2]$, $[-2, -1]$, $[-1, 0]$, $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, we obtain the following table:

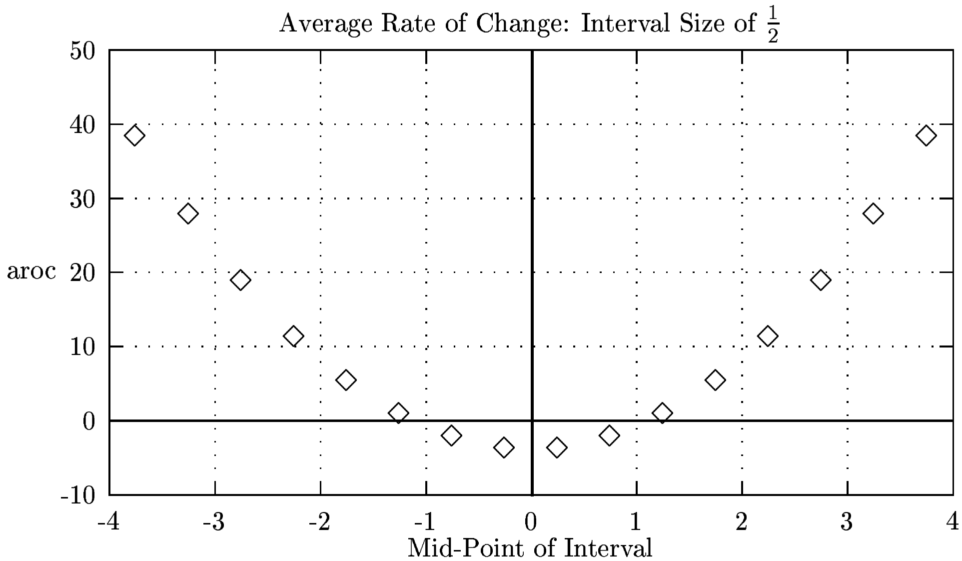
Interval	$[-4,-3]$	$[-3,-2]$	$[-2,-1]$	$[-1,0]$	$[0,1]$	$[1,2]$	$[2,3]$	$[3,4]$
Mid-Point of Interval	-3.5	-2.5	-1.5	-0.5	0.5	1.5	2.5	3.5
aroc	33	15	3	-3	-3	3	15	33

If we want a graph of the data above, a reasonable way to represent the situation would be to plot the aroc of each interval against the midpoint of the interval. We will regard an aroc function as a function that uses midpoints of intervals as inputs and the aroc's on the corresponding intervals as outputs. (We could define such a function as sending intervals to aroc's, but this would take us into new territory, since then the inputs would not be numbers.)

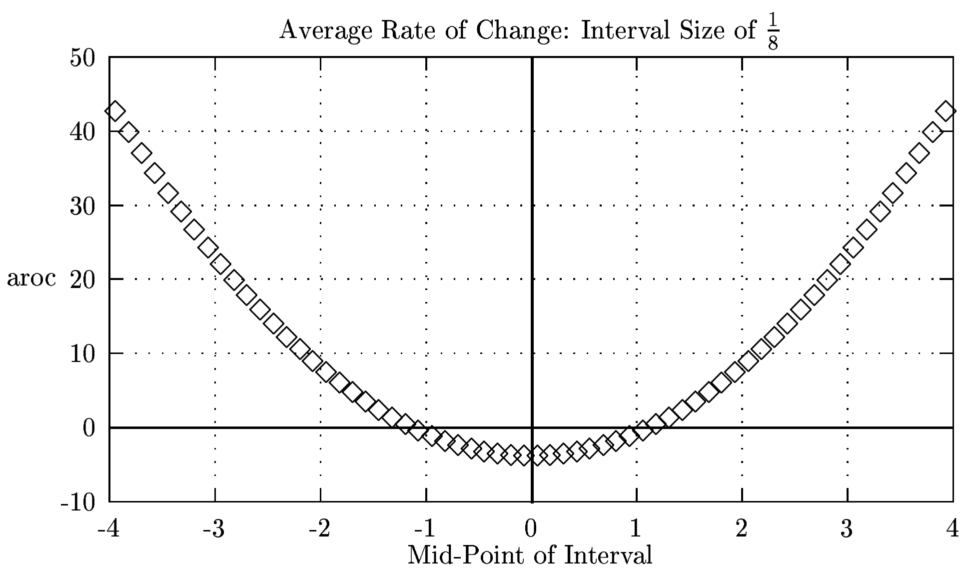


A pattern appears!

The aroc function graphed above is not the only possible aroc function for f. Specifically we could have used intervals of a different size, *i.e.*, not length 1. In plotting any aroc function we must be careful to specify the length of the intervals. We will see that as the size of the intervals gets small, the qualitative features of a graph do not change, even though the exact values of the function change slightly. Below is a graph of an aroc function of f for intervals of length 0.5. The intervals plotted include $[-4,-3.5],[-3.5,-3.0],[-3.0,-2.5],\ldots$.



Below is a graph of an aroc function of f for intervals of length 0.25. The intervals plotted include $[-4,-3.75],[-3.75,-3.5],[-3.5,-3.25],\ldots$.



Although the points obtained for shorter intervals do not lie on the same parabola with the points from the longer intervals, it is the case that as shorter and shorter intervals are used, the resulting graph gets closer and closer to a particular parabola.

If you do the exercises on aroc you will see there is a particular type of aroc function associated with all polynomials of any given degree.