MATH 161 — Precalculus[1]
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Math 161 — Chapter 6
Arithmetic of Functions
Information

6.1 Perspective

In the continuing effort to organize our knowledge of different types of functions, we have examined several types of functions; in particular, linear, piecewise linear, quadratic and higher degree polynomials, absolute value and square root functions. We have also gotten new functions from old by translations, reflections and stretching. Now we go on to create new functions by combining old ones in various ways. We will also analyze functions by breaking them down into combinations of simpler ones, and define the inverse of a function.

6.2 Arithmetic Combinations of Functions

Definitions and Examples

Addition, subtraction, multiplication and division of functions are defined straightforwardly.

Definition: Given two functions f and g with overlapping domains, we define new functions $f+g$, $f-g$, fg, and f/g for all values x in the domain of both f and g as follows:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(fg)(x) = f(x)g(x)$
4. $(f/g)(x) = f(x)/g(x)$ as long as $g(x) \neq 0$ (since we can't divide by 0)

Our first examples use two "baby" functions defined as follows:

$$d = \{(1,2), (3,4), (5,6)\}$$

and

$$e = \{(1,5), (3,-4), (5,0)\}.$$

Example 1: We can add the outputs of the two functions for the single input value $x = 1$ and obtain

$$(d + e)(1) = d(1) + e(1) = 7.$$

Example 2: We can add the two functions as a whole, *i.e.*, add the outputs for each input value, and get

$$d + e = \{(1, 2 + 5), (3, 4 + (-4)), (6 + 0)\} = \{(1, 7), (3, 0), (5, 6)\}.$$

Example 3: The difference of the two functions d and e is :

$$d - e = \{(1, -3), (3, 8), (5, 6)\}.$$

Example 4: The product of the two functions d and e is:

$$de = \{(1, 10), (3, -16), (5, 0)\}.$$

Example 5: We can find the quotient of the two functions:

$$d/e = \left\{\left(1, \frac{2}{5}\right), (3, -1)\right\}.$$

Note that d/e is not defined for $x = 5$ since $\frac{6}{0}$ is not defined.

Now consider the two functions

$$f(x) = x^2$$

and

$$g(x) = x + 2.$$

The following examples show a variety of arithmetic combinations of f and g for the specific input value $x = 5$, and also show the domains of the combinations.

Example 6: $(f + g)(5) = f(5) + g(5) = 25 + 7 = 32$ The domain of $(f + g)$ is all real numbers.

Example 7: $(f-g)(5) = f(5) - g(5) = 25 - 7 = 18$ The domain of $(f-g)$ is all real numbers.

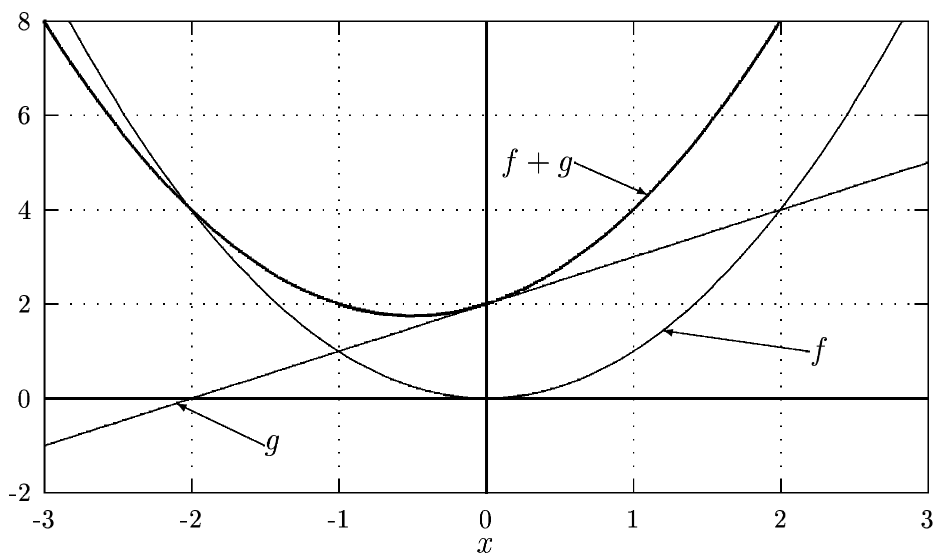
Example 8: $(fg)(5) = f(5)g(5) = 25 \times 7 = 175$ The domain of (fg) is all real numbers.

Example 9: $(f/g)(5) = f(5)/g(5) = 25/7 = 3\frac{4}{7}$ The domain of (f/g) is all real numbers *except* $x = -2$.

Example 10: A general expression for the sum of f and g valid for a general input x is

$$(f+g)(x) = x^2 + x + 2.$$

The graphs of f, g and $f+g$ are shown below on the same coordinate system.



The definitions of the arithmetic operations on functions are so simple they appear to be just new ways of describing how to add, subtract, multiply or divide the value of two functions at a given point. It's fine to think of them that way for purposes of computation. However, it's worth noting that there is a new idea here. We are being asked to think of functions as having a life of their own that enables them to combine, and not just at a particular value of interest, but over the entire domain they have in common. It's a different perspective, something like thinking of a route to school (or anywhere) as a thing in itself, rather than as the sum of the steps and turns taken.

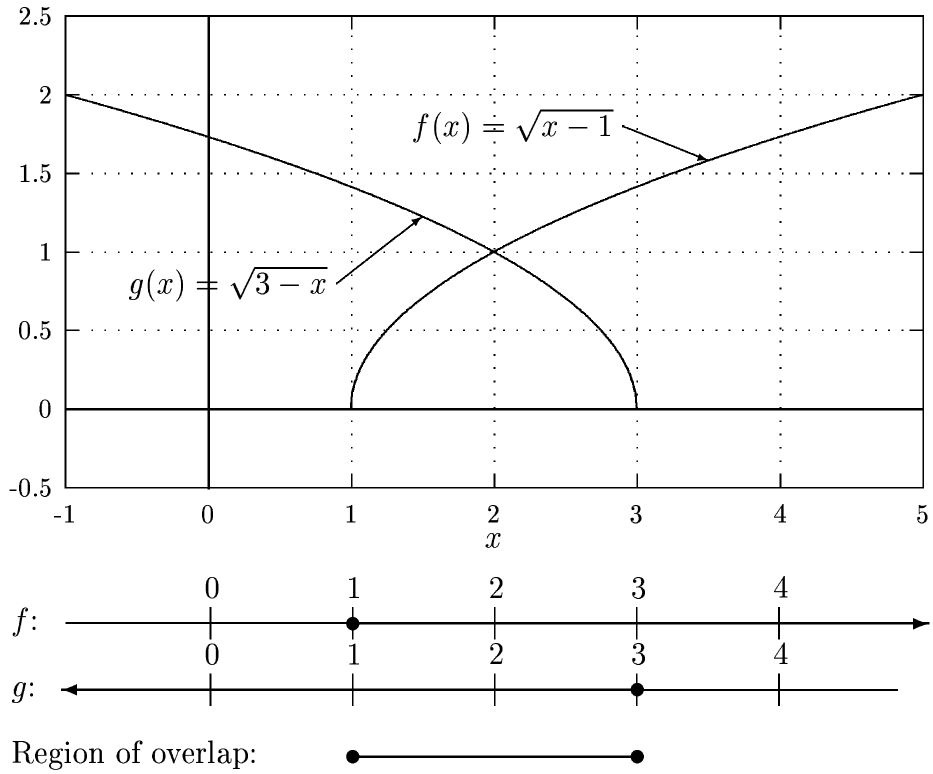
Domains of Arithmetic Combinations of Functions

Example 11: Consider the domains and ranges of the functions d and e above as well as the functions obtained from them. Note that the domain of all the new functions except d/e is the same as the domain of d and e: $\{1, 3, 5\}$. The domain of d/e is $\{1, 3\}$. We cannot use 5 as an input because for 5 the rule would require division by 0.

As stated in the definition the arithmetic combinations of two functions are defined on the set where both functions are defined *i.e.*, on the intersection of their domains. Many functions, including linear, quadratic, higher-degree polynomial and absolute-value functions, are defined for all real numbers, so their arithmetic combinations are too, except that in the case of division, we must exclude values that make the denominator 0, as we did with d/e. In general, if we are dividing by a function $g(x)$, we must exclude the values of x that make $g(x) = 0$.

Example 12: Let $f(x) = 2x+1$ and $g(x) = \sqrt{x}$. The domain of f is all real numbers, and the domain of g is $x \geq 0$. So the common domain (where both are defined) is the set of x such that $x \geq 0$.

Example 13: Let $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{3-x}$. The domain of f is the set of all x such that $x \geq 1$, and the domain of g is the set of all x such that $x \leq 3$. (Recall that the domains are the set of values for which the expressions under the radical are greater than or equal to 0.) So the common domain is the set of all x such that $x \geq 1$ and $x \leq 3$. This is generally written $\{x : 1 \leq x \leq 3\}$, or just $1 \leq x \leq 3$. We can also write the interval: $[1, 3]$. A graph can be helpful in determining the domain. It is useful to sketch the domains of the two original functions on the same number line (or on two number lines, one directly below the other), and look for the overlap, as shown below.



Example 14: For the same functions as in the previous example, the domain of f/g would be the interval $[1, 3)$. The value 3 must be excluded from the domain because it would make the denominator $(f(3)/g(3))$ equal to zero, and we cannot divide by zero.

6.3 Composition of Functions

Another way of combining two functions is by *composition*. The composition of a function $f(x)$ with the function $g(x)$ applies f to the outputs of g:

Definition: The composition of $f(x)$ with $g(x)$ is defined to be $f \circ g(x) = f(g(x))$. The domain of $f \circ g(x)$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Thus to find $f \circ g(x)$, first find $g(x)$, then f of the result. Note that for composition, order matters, as with subtraction or division.

Example 15: Let $f = \{(0,4),(1,8),(3,6),(4,7),(5,8)\}$ and $g = \{(1,-1),(2,0),(3,5)\}$. Then $f \circ g = \{(2,4),(3,8)\}$. Let's go through this step by step: We start with g. $g(1)$ is -1, but -1 is not in the domain of f, so we can't use it. The next input for g is 2, and $g(2) = 0$. Also, $f(0)$ is defined to be 4, so $f \circ g(2) = 4$. The other input of g is 3: $g(3) = 5$ and $f(5) = 8$, so $f \circ g(3) = 8$. The domain of $f \circ g$ is $\{2,3\}$.

Example 16: Let $f(x) = 2x$ and $g(x) = x + 3$. Then $f \circ g(5) = f(g(5))$. Since $g(5) = 8$, we have $f(g(5)) = f(8) = 16$. Diagrammatically we have:

$$5 \xrightarrow{g} 8 \xrightarrow{f} 16$$

Example 17: As above, let $f(x) = 2x$ and $g(x) = x + 3$. This time, let's find the composition $g \circ f(5)$. In this case, we first find $f(5) = 10$. Then we find $g(10) = 10 + 3 = 13$. Note that $g \circ f(x) \neq f \circ g(x)$.

$$5 \xrightarrow{f} 10 \xrightarrow{g} 13$$

Example 18: Once again, let $f(x) = 2x$ and $g(x) = x + 3$. Let us find the general formula for $f \circ g(x)$. Since $g(x) = x + 3$, we have that $f \circ g(x) = f(g(x)) = f(x+3) = 2(x+3) = 2x + 6$.

$$x \xrightarrow{g} (x+3) \xrightarrow{g} 2(x+3)$$

The new function obtained by taking the composition of two functions may be given a name of its own; for example the composition $g \circ f$ may be denoted by the letter h, and we may write $h(x) = g \circ f(x)$. If $f(x) = 2x$ and $g(x) = x + 3$, then $h(3) = g \circ f(3) = g(6) = 9$. The general formula for h is $h(x) = g \circ f(x) = g(f(x)) = g(2x) = 2x + 3$.

6.4 Decomposition of Functions

In certain situation in calculus it is useful to reverse the process of composition; *i.e.*, to break a function down into a composition of simpler functions, in order to apply rules given in terms of the simpler functions. This process is called *decomposition*. To do it, observe what operations are performed on the variable, and in what order. Then write the functions that describe each operation. (For example, if the first operation is squaring the variable, let the first function be $f(x) = x^2$.) Then write the composition of the simpler functions in the correct order to produce the original function.

Example 19: Write the function $h(x) = x^2 + 1$ as a composition $g \circ f$ of two simpler functions g and f.
Solution: To evaluate $h(x)$ at particular values of x we first square x, then add one to the result. So the first function (written closer to the variable in the composition) is $f(x) = x^2$. For the second we write $g(x) = x + 1$. The composition $g \circ f(x)$ is $h(x)$.

Sometimes there is choice of how to break down a function, and sometimes a function breaks down into a composition of more than two simpler functions.

Example 20: Let $h(x) = \frac{1}{2x}$. We may decompose h as $g \circ f$ where $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{2}x$, or as $g \circ f$ where $f(x) = 2x$ and $g(x) = \frac{1}{x}$.

Example 21: Write $p(x) = (\sqrt{x} + 4)^3$ as a composition of simpler functions.
Solution: First analyze the function $p(x)$. In applying p to x we first take the square root of x, then add 4 to the result, and then cube that result. So if we let

$$f(x) = \sqrt{x} \qquad g(x) = x + 4 \qquad h(x) = x^3$$

then

$$p(x) = h \circ g \circ f(x)$$

Checking this,we first take$f(x) = \sqrt{x}$, then $g(\sqrt{x}) = \sqrt{x} + 4$, then $h(\sqrt{x} + 4) = (\sqrt{x} + 4)^3$.

Note: not every complicated function can be expressed as a composition of simpler functions. If the original function can be described in terms of a sequence of steps done in a particular order, starting with the variable, doing something to it, then something else to the result, something else to that result, etc., then the original function can be regarded as a composition. However, if you have to do two different things to the input, then combine the results, then composition isn't helpful.

Example 22: Let $f(x) = \sqrt{x} + x^2$. This function doesn't lend itself to decomposition. Neither does $g(x) = (x-3)/(x-5)$. In each case two different operations are performed on the variable, then the results are combined.

6.5 Inverse Functions

With some functions it's possible to work backwards, in the sense that for a particular output it's possible to find what the input was. For example, consider the function

$$f(x) = 2x - 3.$$

Suppose you know that the output is 7, and you want to know the input it came from. You can get this by inspection: 5. You have just found that $f^{-1}(7) = 5$.

Example 23: Imagine that you have a partner in class who tells you that she has chosen a value of x as an input to the function f defined in the paragraph above, and that the output of the function is 7. You are then challenged to determine the value of the input x that makes $f = 7$.
Solution: In this case it is probably easy to deduce from inspection that if $x = 5$, then $f(5) = 2 \times 5 - 3 = 7$, as desired. Or, set $7 = 2x - 3$ and solve for x.

If you need to figure out very many inputs to a function when given only outputs, it behooves you to find a mathematical rule that allows you to do this as easily as possible. Suppose you are given a one-to-one function $f(x)$ and want to find a formula for its inverse. Do the following:

- Replace $f(x)$ by y (to simplify appearances).
- Replace each x by y and each y by x (for reasons that may become apparent when you do the class exercises).
- Solve for the new y (if possible). (If not, you can't get an inverse algebraically, and will have to settle for graphical approximation.)
- Replace the new y by $f^{-1}(x)$.

Note: The exchange of x and y occurs because outputs of the original function f will be inputs of f^{-1}. For any point (x, y) on the graph of f, the point (y, x) is on the graph of f^{-1}. But we prefer to keep x as the standard symbol for the input and y for the output, even for inverse functions. It may be a little confusing at first. There is no way around this.

Example 24: Find the formula for the inverse of the function $f(x) = 2x - 3$.
Solution:

- Replace $f(x)$ by y:
$$y = 2x - 3$$
- Interchange x and y:
$$x = 2y - 3$$
- Solve for the new y:
$$y = \frac{x+3}{2}$$
- Replace the new y by $f^{-1}(x)$:
$$f^{-1}(x) = \frac{x+3}{2}$$

Definition: Given a function f, the *inverse* of f, if it exists, is denoted by f^{-1}, and is the function with the property that $f^{-1} \circ f(x) = x$; i.e., applying f to x, and then f^{-1} to the result, gives the original number x back again. This can be represented by the following diagram:

$$x \xrightarrow{f} f(x) \xrightarrow{f^{-1}} x$$

Example 25: Let $f = \{(1,0),(2,7),(3,4)\}$. Then $f^{-1} = \{(0,1),(7,2),(4,3)\}$.

Now suppose you and a class partner are working with the function

$$g(x) = x^2.$$

You secretly choose an input value of x, and announce that you got an output of 4 from your input. Can your partner deduce your input? Be careful! Your classmate can't say for sure: it might be $+2$, or it might be -2. You don't have an inverse function because you have a choice of answers and that violates the definition of function. If we want a situation in which we do have an inverse function, we have to rule out choice. In this case, we can do so by restricting the domain of the original function so as to allow only numbers greater than or equal to zero. In order for function to have an inverse, it must be a one-to-one function.

Definition: A **one-to-one function** is a function such that each element of the range has exactly one element of the domain assigned to it.

A one-to-one function has a unique inverse. A function which is not one-to-one does not. But because it is often useful to have an inverse, a function is sometimes modified by restricting the domain to a set on which it is one-to-one. In the example above, restricting the domain of the function g to the set of non-negative real numbers, written $\{x|x \geq 0\}$, has the effect of guaranteeing that each element of the range has only one domain element assigned to it. Once this is done, g^{-1} is well-defined.

Example 26: Give a rule for the inverse of g under the assumption that the domain is $\{x|x \geq 0\}$. Give a rule for the inverse of g under the assumption that the domain of g is $\{x|x \leq 0\}$.
Solution: To find the value of the input x you must take the square root of the value of $g(x)$. A positive number has two square roots, one positive and one negative. If we choose the domain $\{x|x \geq 0\}$, then we use the positive square root for the inverse; if $\{x|x \leq 0\}$ we use the negative square root. (If $g(x) = 0$, the inverse of g is zero.)

Example 27: Let $f = \{(0,1),(2,3),(5,1)\}$. The function f is not one-to-one, since $f(0) = f(5) = 1$. If we restrict the domain of f to $\{0,2\}$, then f^{-1} is defined: $f^{-1} = \{(1,0),(3,2)\}$. If we restrict the domain to $\{2,5\}$ then $f^{-1} = \{(1,5),(3,2)\}$. It is important to state explicitly what the restricted domain is; this is part of the definition of the function.

Example 28: The relation between the domain and range of a function and its inverse may be seen most easily by looking at a function given by ordered pairs: Let $f = \{(0,1),(2,3),(5,6)\}$. Then $f^{-1} = \{(1,0),(3,2),(6,5)\}$. The domain of f is $\{0,2,5\}$ and this is the range of f^{-1}, since inputs and outputs are interchanged. Similarly, the range of f is $\{1,3,6\}$, and this is the domain of f^{-1}.

For any f with an inverse f^{-1}, the range of f is the domain of f^{-1}, and the domain of f is the range of f^{-1}. For the functions with which we usually work, we determine whether the function is one-to-one by graphical or algebraic means, and determine how to restrict the domain by the same methods.

There are a variety of situations in which it is useful to know what input produced a particular output, and you will encounter situations in calculus where it is necessary to do this.

For some of the exercises in this chapter you will need to use higher order roots or fractional exponents. See the algebra pushups if you want to review.