MATH 161 — Precalculus[1]
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Math 161 — Chapter 7
Rational Functions
Rational Functions

7.1 Introduction

We have already developed new functions from old using translations, stretching, arithmetic operations on functions, composition and inverses. We continue in this direction, building from polynomials. When we add, subtract or multiply polynomial functions, we get other polynomial functions. But when we divide one polynomial by another we get a new type of function with characteristics that we have not yet seen.

We can compare the situation to that with numbers: we start with natural numbers (1, 2, 3, , *etc.*) and perform arithmetic operations on them: $+, -, \times$, and $\div$. If we add or multiply two natural numbers, the answer is also a natural number. When we subtract a larger natural number from a smaller, we get a new kind of number, a negative number. We expand the number system to include the newcomers. The new system is the set of integers $(\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots)$. When we divide an integer by another, we usually do not get an integer; we get a fraction. We expand the number system to include these. The new system is the set of rational numbers. However, we can't expand the system to include all divisions: division by zero doesn't work. We don't divide by zero.

We now consider the type of function we get when we divide one polynomial by another.

7.2 Definition and examples of rational functions

We define an expression which is the quotient of one polynomial by another as a *rational expression*, and a function whose formula can be expressed as a rational expression as a *rational function.* (As in the case of numbers, here the term *rational* comes from the word *ratio*; we are taking the ratio of two polynomials.)

Definition: A rational function $r(x)$ is one that can be written in the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials. We do not allow the constant function $y = 0$ as a denominator for a rational function. So $q(x)$ is not identically zero (*i.e.,* not always zero).

However, we do allow in the denominator polynomials which have zeroes. The domain of a rational function is assumed to be all real numbers except those that make the denominator zero. An important question to consider when dealing with a particular rational function is what numbers these are.

Example 1: $$r(x) = \frac{x+3}{2x-1}$$

We must exclude from the domain values of x such that $2x - 1 = 0$; that is, we must exclude the value $x = \frac{1}{2}$. The domain is all real numbers except $\frac{1}{2}$; *i*.e., $x|x \neq \frac{1}{2}$.

Example 2: $$r(x) = \frac{1}{x}$$

We must exclude 0 from the domain.

Example 3: $$r(x) = \frac{x^3 + 5x^2 - 7x + 2}{4x^5 - 3x^4 + x^2 - 9}$$

The values that must be excluded from the domain are the zeros of $4x^5 - 3x^4 + x^2 - 9$ This polynomial does not factor, and we must approximate its roots using technology. Recall a fifth degree polynomial may have as many as five roots, and must have at least one. This polynomial has only one root, approximately 1.3292. Its domain is all real numbers except this one.

Example 4: $$r(x) = \frac{x^3}{x^2 + 1}$$

Since the polynomial $x^2 + 1$ has no roots, no values need to be excluded from the domain, and the domain of this rational function is the set of all real numbers, or $(-\infty, \infty)$.

Example 5: $$r(t) = \frac{(t-9)(t+3)(t-2)}{(t+8)(t^2 - 2t + 3)}$$

Since $x + 8$ is a factor of the denominator, -8 is a root of it. We must determine whether the quadratic factor of the denominator has any roots. Taking the discriminant of $(t + 8)(t^2 - 2t + 3)$, we find that it does not. So the domain of this rational function is the set of all real numbers except -8.

Example 6: $g(x) = x^2 + 3x + 7$

A polynomial is a special case of a rational function, since taking the denominator to be the constant function $q(x) = 1$, we can regard it as

$$g(x) = \frac{x^2 + 3x + 7}{1}$$

The domain consists of all real numbers.

Example 7: $r(x) = 8 + \frac{1}{x}$

This is a rational function because it can be put in the form $\frac{p(x)}{q(x)}$: we use x as a common denominator and write

$$r(x) = \frac{8x + 1}{x}$$

The domain consists of all real numbers except 0.

7.3 Roots and y-intercept of a rational function

The roots of a rational function in standard from are the roots of its numerator, and finding them is thus a matter of finding the roots of that polynomial. The y-intercept of a rational function is found as with any function, by substituting 0 for x. With a rational function, it may happen that doing so gives a denominator of 0, in which case the function is undefined at zero and does not have a y-intercept. Example 7 is an illustration of this.

7.4 The Behavior of a Rational Function as $x \to \pm\infty$

Recall what we saw with polynomials as $x \to \pm\infty$: the term of highest degree got so much larger than the other terms that they could be regarded as small change, and the size of the polynomial was well approximated by the term of highest degree. This always happens eventually even if the lower-degree terms have much larger coefficients than the term of highest degree, and even if the lower-degree terms have coefficients of opposite sign to that of the highest-degree term.

In the case of rational functions, to see what happens as $x \to \pm\infty$, it is useful to compare the degree of the polynomial in the denominator to that of the polynomial in the numerator. There are three cases:

1. The degree of the polynomial in the denominator is *larger* than the degree of the polynomial in the numerator

   Then as $x \to \pm\infty$ the absolute value of the denominator eventually gets much larger than the numerator in absolute value, so we are looking at a fraction of the type $\frac{large}{muchlarger}$. The value of such a fraction is close to zero (like a million over a trillion). So

   $$\lim_{x \to \pm\infty} r(x) = 0$$

   Example 8: Let

   $$r(x) = \frac{x-1}{x^2+2}$$

   The denominator has higher degree than the numerator, and for large values of x the absolute value of the denominator is therefore larger than that of the numerator, and the resulting fraction is small. For example, $r(10) = 9/102 \approx 0.098$, $r(100) = 91/10002 \approx 0.009$, $r(1000) = 998/1000002 \approx 0.000998$.

2. The degree of the polynomial in the denominator is *smaller* than the degree of the one in the numerator.

   In this case, as $x \to \pm\infty$ the denominator, while it continues to get large, doesn't get nearly as large as the numerator. So we are looking at a fraction of the type $\frac{very large}{large}$. The value of such a fraction is large (like a trillion/million). So

$$\lim_{x \to \pm\infty} r(x) = \pm\infty$$

To determine whether the limit of $r(x)$ as $x \to \infty$ is ∞ or $-\infty$ for any given $r(x)$, determine the sign of the polynomial in the numerator and that of the polynomial in the denominator as $x \to \infty$ to see whether they have the same or opposite signs. If the signs are the same, the limit is ∞; if not, $-\infty$. Apply the same method when $x \to \pm\infty$.

Example 9: Let

$$r(x) = \frac{x^2 - 1}{2 - x}$$

In this case the numerator has higher degree than the denominator, and $lim_{x \to \infty} r(x) = -\infty$, $lim_{x \to -\infty} r(x) = \infty$ A few sample values: $r(10) = 99/(-8) = -12.375$, $r(100) = 9999/(-98) \approx -102$, $r(1000000) = 999999999999/(-999998) \approx -1000002$; $r(-10) = 99/(-12) = 8.25$, $r(-100) = 9999/102 \approx 98$, $r(-1000000) = 999999999999/1000002 \approx 999998$.

3. If the degrees of the two polynomials are *the same*, then we again use the fact that as $x \to \pm\infty$ the highest-degree term in each polynomial eventually gets so much larger than the lower-degree terms that we can disregard the lower-degree terms. Let's write out the polynomials in general form:

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \ldots + b_2 x^2 + b_1 x + b_0}$$

Since as $x \to \pm\infty$ we can get a good approximation to the size of each polynomial from its first term alone, we have that

$$r(x) \approx \frac{a_n x^n}{b_n x^n}$$

But in the fraction on the right we can cancel the x^n's, and see that $r(x) \approx \frac{a_n}{b_n}$. Thus

$$\lim_{x \to \pm\infty} r(x) = \frac{a_n}{b_n}$$

Several more examples of the various cases:

Example 10: $$\lim_{x\to\infty} \frac{7x^2 - 3x}{2x^3} = 0$$

Example 11: $$\lim_{x\to\infty} \frac{x^7 - 6x^4 + 2x^3 - x + 4}{x^5 - x^2 + 3x} = \infty$$

Example 12: $$\lim_{x\to\infty} \frac{-x^8}{x^3 - 2} = -\infty$$

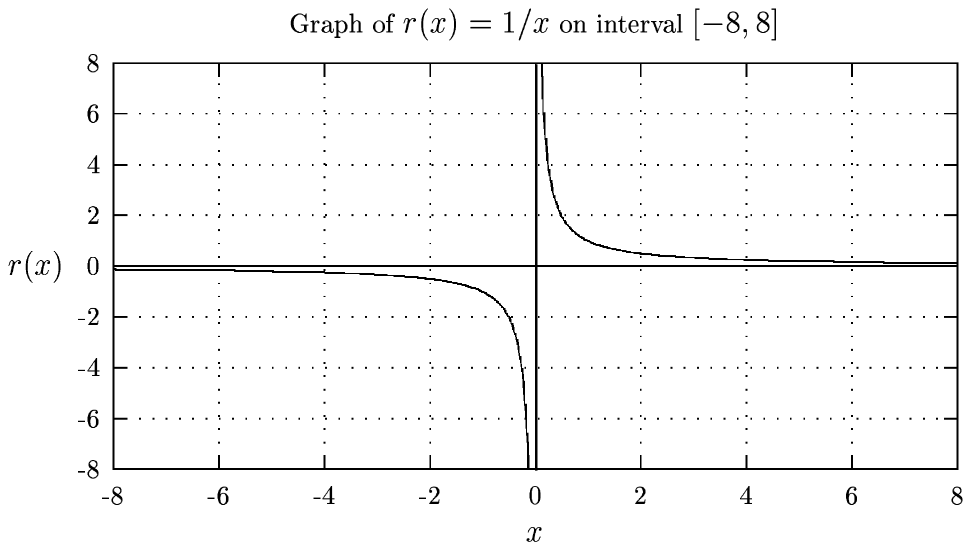
Example 13: $$\lim_{x\to\infty} \frac{4x^3 - 1000x^2 + 0.01x + 3}{7x^3 + 50000x^2 - 90x + 14} = \frac{4}{7}$$

Example 14: $$\lim_{x\to\infty} \frac{5}{x} + 3 = \frac{5 + 3x}{x} = 3$$

In this case the rational function was converted to standard form. We could also find this limit by observing that the function is a translation 3 units up of the function $f(x) = \frac{5}{x}$, which has limit 0 as $x \to \infty$. This method is quicker.

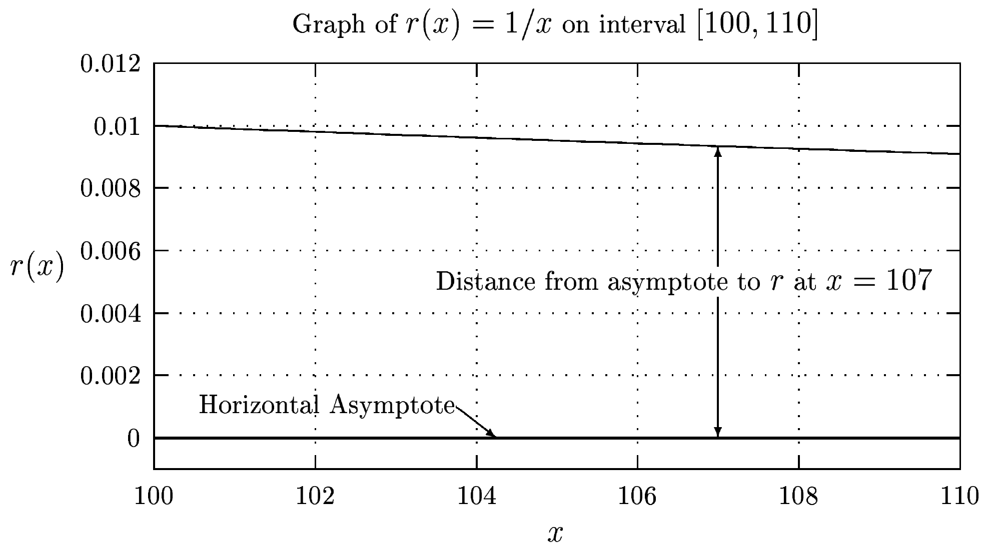
7.5 Asymptotes

A *horizontal asymptote* of a function $f(x)$ is a line (or in fancy cases a curve, but we ignore this possibility) with the property that, as x or y approaches $\pm\infty$ the graph of the function gets close to the line but doesn't intersect it. The graph of the function appears to run alongside the asymptote, getting closer all the time but never touching it. We see an example of this with the function $r(x) = \dfrac{1}{x}$. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, the x-axis is an asymptote to the graph of the function $r(x) = \frac{1}{x} = 0$; the further x is from zero, the closer the graph of the curve gets to the x-axis.



We will pay particular attention to *horizontal* and *vertical* asymptotes. The graph above has the line $y = 0$ (*i.e.* the x-axis) as a horizontal asymptote because it gets close to the x-axis as x gets large.

When we speak of the graph getting closer to a horizontal asymptote we mean that as x gets larger the perpendicular distance from $(x, f(x))$ to the asymptote gets smaller. This is the vertical distance between the point and the x-axis, of course, which equals $|f(x)|$ For example, the distance of $r(x) = 1/x$ from the x-axis for $x = 100$ is $1/100$ — this is the height of the function value above the x-axis. For larger values of x the distance to the x-axis is even smaller; the graph is within $1/100$ of the x-axis for all x greater than 100.



The distance of a graph from any asymptote is defined similarly. An asymptote may be vertical; for example, as x gets close to 0, the graph of $r(x)$ gets ever closer to the y-axis. The y-axis is a vertical asymptote of $r(x)$.

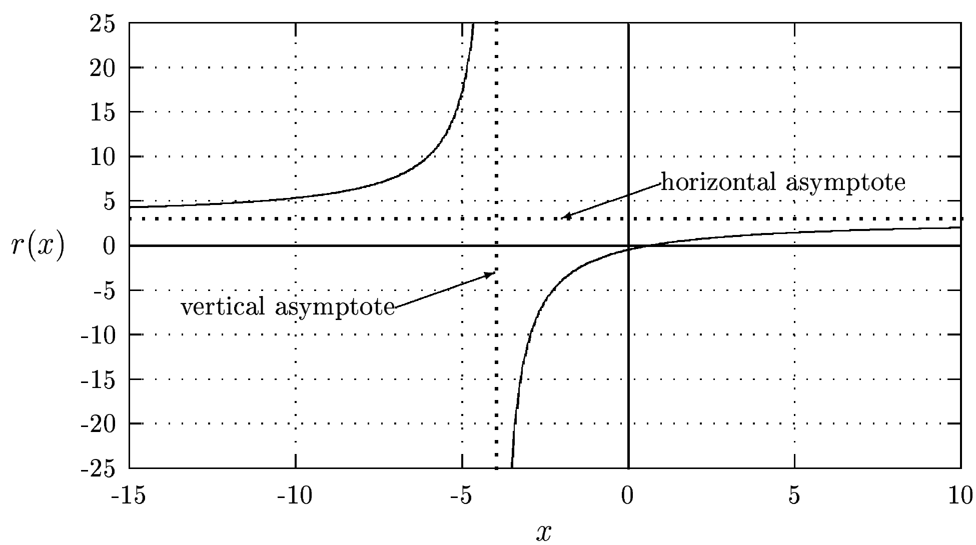
Note that asymptotes present a special graphing issue: no matter what scale we use, before long the graph of the function appears to merge with the asymptote. This is just a property of the particular graph; in the world of abstract mathematics the graph never actually touches the asymptote. But it gets so close in the picture that you can't see the difference. When sketching a graph, cut off the sketch before the apparent merging point. When using a graphing calculator or computer you usually have to play around with the window to get something suitable.

To find the vertical asymptotes of a rational function, find the zeroes of its denominator. (These are the values of x that are excluded from the domain.) Unless such a zero is also a zero of the polynomial in the numerator (in which case more analysis is needed–this is discussed below) the vertical line through the zero is a vertical asymptote.

To find the horizontal asymptotes of a rational function, we need to find the limit of the function as $x \to \pm\infty$, as described earlier. If the value of the rational function approaches a finite number c as $x \to \infty$, the line $y = c$ is a horizontal asymptote to the graph of the function. If the value of the function approaches $\pm\infty$ as $x \to \pm\infty$, then the function does not have a horizontal asymptote.

Example 15: Find the vertical and horizontal asymptotes, if any, to the graph of $r(x) = \dfrac{3x-2}{x+4}$.

Solution: The value -4 is not in the domain of the function, and the function has the vertical asymptote with equation $x = -4$. The limit of this function as $x \to \pm\infty$ is 3, and the line $y = 3$ is a horizontal asymptote. The graph of the function is shown below with its horizontal and vertical asymptotes.



A rational function $r(x) = p(x)/q(x)$ does not always have a vertical asymptote at a value of x_0 for which the denominator $q(x_0) = 0$. There is an asymptote if the denominator is zero at x_0 and the numerator isn't. If the polynomial q of the denominator is zero at x_0, then, as you may recall from work with polynomials, $(x - x_0)$ is a factor of q. Suppose $x - x_0$ is also a factor of the numerator $p(x)$. If the factor occurs to the same power in numerator and denominator, then this factor can be canceled out, the resulting expression is well behaved at $x = x_0$, and $r(x)$ does not have an asymptote there. It's just missing a point there because $x0$ is not in the domain. (It doesn't come up often, but if the factor occurs to a higher power in the denominator than in the numerator, then not all the factors cancel out, and there is an asymptote.)

Example 16: Let

$$f(x) = \frac{(x+1)(x-1)}{(x-1)(x-2)}.$$

The domain of $f(x)$ is all real numbers except 1 and 2. At every point in its domain, $f(x) = (x+1)/(x-2)$. Using this formula at $x = 1$ we get $f(1) = -2$. Since 1 is not in the domain, the graph of f has a point missing where $(1, -2)$ would be. Note that we don't add the point back into the domain — we stick with the original definition of f for this. Also note that no graphing device is going to show that a point is missing–you have to realize it for yourself.

7.6 One-sided limits

Recall the function $r(x) = (3x-2)/(x+4)$ of Example 15. This function has an asymptote at $x = -4$. On the left side $(x < -4)$ $r(x)$ gets very large as x gets close to -4, and on the right $(x > -4)$ it gets very large in absolute value, but negative. In order to describe the situation we introduce the following notation.

Definition: If $f(x)$ gets large without bound as x gets close to x_0 with x to the left of x_0 , we write

$$\lim_{x \to x_0^-} f(x) = \infty.$$

This is read "the limit of $f(x)$ as x approaches x_0 from the left is positive infinity." The minus sign above and to the right of x_0 indicates x approaches x_0 from values slightly less than x_0. Similarly, if $f(x)$ gets large in absolute value, but negative as x gets close to x_0 and x is to the left of x_0 , we write:

$$\lim_{x \to x_0^-} f(x) = -\infty.$$

If x approaches x_0 from the right instead of the left, and $f(x)$ gets large without bound, we write

$$\lim_{x \to x_0^+} f(x) = \infty,$$

and if $f(x)$ gets large in absolute value, but negative

$$\lim_{x \to x_0^+} f(x) = -\infty.$$

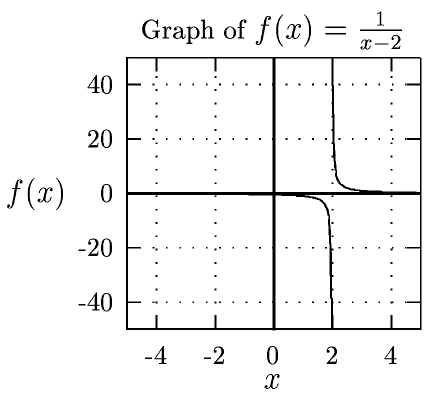
Example 17: For the function $r(x) = (3x-2)/(x+4)$,

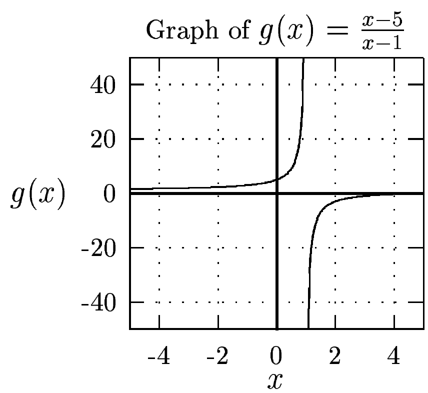
$$\lim_{x\to 4^-} r(x) = +\infty \quad \text{and} \quad \lim_{x\to 4^+} r(x) = -\infty$$

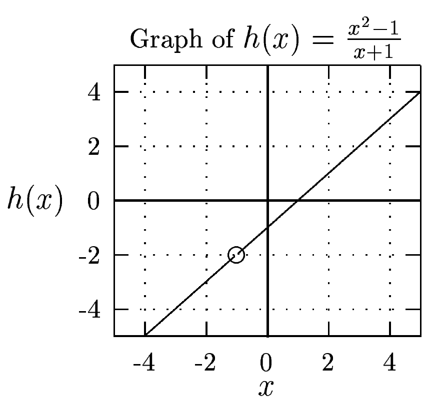
Definition: A value x at which a rational function r gets large without bound in absolute value is called a *pole*. Thus a pole is a value of x at which the function has a vertical asymptote.

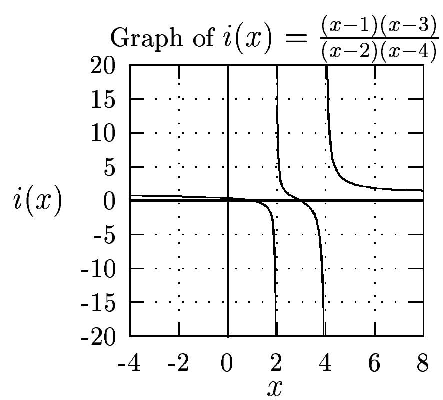
7.7 Graphs of rational functions

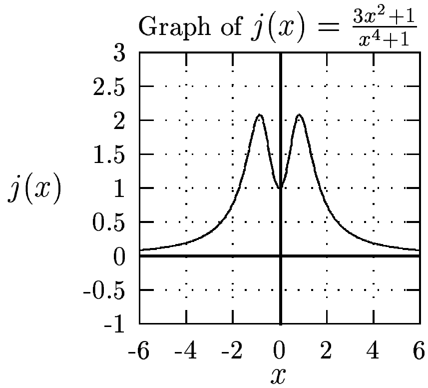
The graphs below illustrate some of the variety that occurs in rational functions.



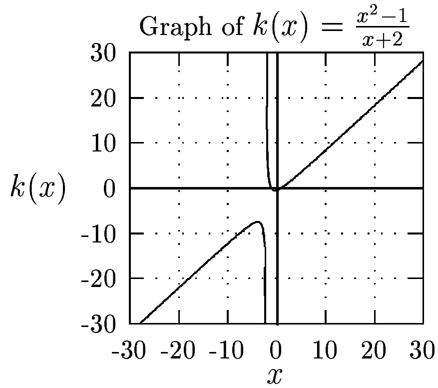








Graph of $j(x) = \frac{3x^2+1}{x^4+1}$
$j(x)$
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-6 -4 -2 0 2 4 6
x



Graph of $k(x) = \frac{x^2-1}{x+2}$
$k(x)$
30
20
10
0
-10
-20
-30
-30 -20 -10 0 10 20 30
x