MATH 161 — Precalculus[1]
Community College of Philadelphia

Joanne Darken
Martin Ligare

[1]Materials produced with the support of the National Science Foundation through a grant to the Middle Atlantic Consortium for Mathematics and its Applications throughout the Curriculum (MACMATC).

Math 161 — Chapter 8
Conic Sections
Information

8.1 Introduction

Distances are important in mathematics as in life, and within a coordinate system we can calculate precisely the distance between two points. In this chapter we recall the formulas for finding the distance between two points on a line and in a two–dimensional coordinate system, and we define the distance between a point and a line. We apply this knowledge to the examination of conic sections, a collection of curves of types that arise in nature as trajectories of planets other celestial bodies and in fact of any object in a gravitational field. These curves can all be described simply in terms of distances.

8.2 Distances

Distance between two points on the number line

As you may recall, the absolute value of a number or an expression may be thought of as its distance from zero on the number line, disregarding direction, and the absolute value of the difference of two numbers a and b is the distance between the points representing them on the number line. For example, $|5-3| = 2 = |3-5|$, and 2 is the distance between the points 3 and 5 on the number line.

Distance between two points in the plane

Now we consider the distance between two points in a Cartesian plane. If two points lie on a horizontal line then they have the same y-coordinate and the distance between them is the absolute value of the difference of their x-coordinates. For example, $(4,2)$ and $(7,2)$ lie on the horizontal line $y = 2$ and the distance between them is $|7-4| = 3$.

The distance between two points (x_1, y) and (x_2, y) on the same horizontal line is $|x_2 - x_1|$ Similarly, if two points lie on a vertical line then they have the same x-coordinate and the distance between them is the absolute value of the difference of their y-coordinates. For example, $(7, 2)$ and $(7, 6)$ both lie on the vertical line $x = 7$, and the distance between them is $|6 - 2| = 4$.

Similarly, the distance between two points (x, y_1) and (x, y_2) on the same vertical line is $|y_2 - y_1|$. To find the distance between any two points in the plane we derive the *distance formula* from the Pythagorean Theorem.

Recall the Pythagorean Theorem: in a right triangle with legs of length a and b and hypotenuse of length c, the sum of the squares of the two sides is the square of the hypotenuse; that is, $a^2 + b^2 = c^2$. This equation can be used to find one of the sides when the other two are known. To find c, for example, we solve the equation for c: $c = \sqrt{a^2 + b^2}$(We use only the positive root since lengths are positive.)

For example, if one leg is of length 3 and the other of length 4, then the hypotenuse must be of length $\sqrt{3^2+4^2} = 5$. The *distance formula* is used to find the distance between two points (x_1, y_1) and (x_2, y_2). It is derived from the Pythagorean Theorem using vertical and horizontal line segments for the legs of a right triangle.

For two points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) respectively, the triangle obtained by this method has sides of length $|x_2 - x_1|$ and $|y_2 - y_1|$ respectively. By the Pythagorean Theorem,

$$\text{Distance}(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is the *distance formula.*

Example 1: Find the distance between $(4, 2)$ and $(7, 6)$.
Solution. The horizontal line through $(4, 2)$, whose equation is $y = 2$, intersects the vertical line through $(7, 6)$, whose equation is $x = 7$, at the point $(7, 2)$, and of course the two are perpendicular. The length of the segment from $(4, 2)$ to $(7, 2)$ is 3, and the length of the segment from $(7, 6)$ to $(7, 2)$ is 4. Thus the segment between $(4, 2)$ and $(7, 6)$ is the hypotenuse of a right triangle with sides 3 and 4, which, as we just saw, has length 5.

Example 2: Let $P_1 = (2, 3)$ and $P_2 = (5, 7)$. Then

$$\text{Distance}(P_1, P_2) = \sqrt{(5-2)^2 + (7-3)^2} = \sqrt{3^2+4^2} = 5.$$

Distance between a point and a line

Consider what you do when you plot a point such as $(3,4)$: you go three units to the right of the origin, then 4 units up. This may be thought of as going 3 units to the right of the y-axis in a direction perpendicular to it, and then 4 units from the x-axis in a direction perpendicular to it. The distance of the point from the y-axis is 3 and from the x-axis is 4.

The distance between any given point and any given line is defined to be the distance between the given point and the point on the line that is closest to it. The segment between these two points is always perpendicular to the line. In the case of horizontal and vertical lines, it is easy to find these distances. For example, the distance of the point $(3,5)$ from the line $y = 7$ is 2, since if we go straight up from the point $(3,5)$ to this horizontal line we hit the point $(3,7)$; this is the point of $y = 7$ closest to $(3,5)$. (See left diagram below.) Similarly, $(3,5)$ is 4 units from the vertical line $x = -1$. (See right diagram below.)

8.3 More on Translations

Recall that given a function $f(x)$, the graph of a function $g(x)$ defined by

$$g(x) = f(x - h) + k$$

is a translation h units horizontally and k units vertically of the graph of f.(See the section in Chapter 5 on translations for illustrations.) Replacing

$g(x)$ by y we write

$$y = f(x - h) + k$$

which we transform to

$$y - k = f(x - h)$$

When we worked only with functions we wanted the function name by itself on the left, so we didn't use the last form above. But when working with equations that do not necessarily represent functions, this is not always necessary (or even possible). For any equation in x and y, if we replace x by $x - h$and y by $y - k$, the graph of the new equation is a translation of the graph of the old h units horizontally and k units vertically. **Example 3:**

Let $y = x^2$. The graph is a parabola with vertex at $(0, 0)$. Replacing y by $y - 3$ and x by $x - 2$ we have $y - 3 = (x - 2)^2$. This could be re-written as $y = (x-2)^2 + 3$, which is the standard equation of a parabola with vertex at $(2, 3)$. **Example 4:** Suppose $x^2 + y^2 - 2x = 8$. Replacing x by $x + 4$, y by

$y - 6$, we have $(x + 4)^2 + (y - 6)^2 - 2(x + 4) = 8$. If we expand and collect like terms,we get $x^2 + y^2 + 6x - 12y + 36 = 0$.

8.4 Conic sections

The conic sections are so called because they are the curves produced when a plane intersects a right circular cone. A right circular cone consists of two ice-cream type cones placed end to end, and extending forever. More precisely, such a cone is produced by choosing a circle, taking a line (its axis) through the center of the circle and perpendicular to the plane of the circle, and a point P on that line not in the plane of the circle. The cone is the set of all points that lie on some line passing through P and a point on the circle. The lines are called *generating lines* of the cone. If the cutting plane is parallel to one of the generating lines of the cone, the curve of intersection is a *parabola*. If it is at right angles to the axis, the result is a circle. If it not at right angles to the axis and cuts only one part of the cone, it is an *ellipse*. If it cuts both parts of the cone it is a *hyperbola*. Below is an illustration of a cone with the various possibilities for planes cutting it. The name of each curve produced is under the cone showing the cut.

You may get a better feeling for these curves if you make a cone, or one part of a cone, from modeling clay (or cheese, or some other readily available substance) and cut it to produce the various curves. We have already worked with parabolas, approaching them from a different point of view–the algebraic. There are algebraic equations for all the conic sections. In fact, any of these curves put in the coordinate plane satisfies an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are real numbers. This is the *general second-degree equation.*

For the parabolas we have worked with, which are functions, we would have $B = C = 0$, $E = -1$, giving the equation $Ax^2 + Dx - y + F = 0$, which we solve for y if we want standard function form.

For example, $f(x) = x^2 + 3x - 8$ could be written in the form of the general equation for a conic as $(-1)x^2 + 0xy + 0y^2 + (-3)x + 1y + 8 = 0$, or $-x^2 - 3x + y + 8 = 0$ (with y replacing $f(x)$).

When we consider the conic sections as plane curves (pulling them down from their three-dimensional cones) we find that each of them can be described as a set of points satisfying specific conditions in terms of distances. (It is surprisingly common in mathematics for objects defined in one situation to appear in a natural way in some other context. It happens with conics, and you will see it in calculus with π.)

Now we consider each of the conic sections in turn as a locus (*i.e.* a set) of points answering a description given in terms of distances.

Circle

A circle is the locus of all points at a given distance from a given point. If the center of the circle is the point with coordinates (h, k) and the radius is r, then a point lies on the circle if and only if it satisfies the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

You can derive this equation by taking the equation that says the distance between (h, k) and (x, y) is r; *viz.*

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

and squaring both sides.

Example 5: The circle of radius 1 centered at the origin has equation

$$x^2 + y^2 = 1.$$

Example 6: The set of all points 5 units from the point $(1, -3)$ is described by the equation

$$(x - 1)^2 + (y + 3)^2 = 25.$$

Expanding and simplifying gives:

$$x^2 + y^2 - 2x + 6y - 15 = 0.$$

Writing this in terms of the general second-degree equation we use $A = C = 1$, $B = 0$, $D = -2$, $E = 6$, $F = 15$.

A compass can be used to construct a circle. The point of the compass is put at the point chosen for the center, and the distance the hinged arm is opened determines the radius. When the other end, with a pencil attached, is touched to paper and swung around, a circle is formed. If you swing an object on a string, its path is a circle (or very close to it–your hand will move a little).

Ellipse

An ellipse is the locus of points the sum of whose distances from two distinct fixed points is constant. Each of the two points is a *focus* of the ellipse (*pl.* foci).

The graph of an ellipse might be described as a squashed circle (but squashed in a very orderly way). It has a long axis, the *major axis*, and a short axis, the *minor axis*. If we use $2a$ as the length of the major axis and $2b$ as the length of the minor axis, then the standard equation of an ellipse with both foci on the x-axis turns out to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the standard equation of an ellipse with both foci on the y-axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

The *center* of an ellipse is the point halfway between its foci, and for any ellipse with an equation of either form above, the center is the origin. For either form, if the non-zero coordinates of the foci are denoted by $\pm c$, the relationship $a^2 - b^2 = c^2$ holds. The sum of the distances of a point (x, y) on an ellipse from each of the two foci is $2a$. The ratio $\frac{c}{a}$ is called the *eccentricity* of an ellipse. It is a measure of the difference between the ellipse and a circle. If the definition of an ellipse did not require the foci to be *distinct* points, so that a circle was a special case of an ellipse, the eccentricity of the circle would be 0. All ellipses according to our definition have eccentricity between 0 and 1. The closer the eccentricity is to 0, the closer the graph of the ellipse is to a circle.

The equations for an ellipse can be derived from the distance formula. For example, if the major axis of the ellipse lies on the x-axis, so that the foci are $(c, 0)$ and $(-c, 0)$ for some c, and (x, y) is a point on the ellipse, the distance between $(c, 0)$ and (x, y) plus the distance between $(-c, 0)$, and (x, y) must add up to $2a$, the chosen total distance. Write the expressions for the two distances, write an equation setting their sum equal to $2a$ (the chosen total distance), struggle through some algebra, and voila!

On the left below is the graph of an ellipse with its foci on the x-axis and on the right one with its foci on the y-axis.

You can construct an ellipse by putting two thumbtacks in a piece of paper on appropriate backing and tying a length of string to them. Then use a pencil or pen to pull the string taut, and move the pencil along the paper, keeping the string taut.

If we re-write the equation of an ellipse with foci on the x-axis in the general form for quadratic equations, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we have $A = b^2$, $B = 0$, $C = a^2$, $D = E = 0$, $F = -1$.

Like a circle, an ellipse cannot be the graph of a function, and must be split into its upper and lower halves to get a functional representation.

The orbits of the planets around the sun are ellipses. The earth's orbit has eccentricity about $1/60$. The sun is one focus, and the major axis is 186 million miles long.

Hyperbola

A hyperbola is the locus of points the difference of whose distances from two fixed points is constant. The fixed points are the *foci*. The standard form of the equation of a hyperbola with center at the origin (halfway between the foci) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

if the foci are on the x-axis, and

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

if the foci are on the y-axis.

If we re-write the equation of a hyperbola with foci on the x-axis in the general form for quadratic equations, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we have $A = \frac{1}{a^2}$, $B = 0$, $C = \frac{-1}{b^2}$, $D = E = 0$, $F = -1$.

The center of a hyperbola is halfway between its foci. Every hyperbola has two branches, and two asymptotes. The asymptotes of the standard hyperbolas are slanted lines. We will not prove this, but the equations of the asymptotes are $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$.

You have actually already worked with hyperbolas, but with a different type of equation, because the foci were not on either axis. The graph of the function $y = 1/x$, for example, is a hyperbola. This is not obvious from its equation, which is quite different from that of a hyperbola with its foci on one of the axes.

Parabola

We have already encountered parabolas as graphs of functions of the type $y = ax^2 + bx + c$. This equation fits into the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Parabolas are conic sections and can be described as loci. A parabola is the locus of all point equidistant from a given point and a given line. The point is called the *vertex* and the line the *directrix* of the parabola.

Summary

The conic sections are wonderful examples of mathematical objects that have simple descriptions in entirely different mathematical contexts, and profound applications to our understanding of the universe. In addition they provide examples and extensions of topics we have already studied, such as symmetry and asymptotes. There is a great deal more to be known about conic sections, and if you would like to pursue your study of them, there are many books and web sites at your disposal.