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Dissonance as a resource for probing the qubit depolarizing channel

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Abstract

Quantum channel identification, a standard problem in quantum metrology, is the task of estimating parameter(s) of a quantum channel. We investigate dissonance (quantum discord in the absence of entanglement) as an aid to quantum channel identification and find evidence for dissonance as a resource for quantum information processing. We consider the specific case of dissonant Bell-diagonal probes of the qubit depolarizing channel, using quantum Fisher information as a measure of statistical information extracted by the probe. In this setting dissonant quantum probes yield more statistical information about the depolarizing probability than do corresponding probes without dissonance and greater dissonance yields greater information. This effect only operates consistently when we control for classical correlation between the probe and its ancilla and the joint and marginal purities of the ancilla and probe.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The precise nature of quantum entanglement and its essential role in quantum information processing remain ongoing subjects of inquiry. Certainly, though, entanglement is a potent and fungible quantum information processing resource [1, 2]. Entanglement can be transformed [3], swapped [4], concentrated [5], catalyzed [6] and distributed over long distances [7], and it plays a central role in such applications as teleportation [8], superdense coding [9] and quantum metrology [10]. Entanglement's remarkable properties stem essentially from the order of correlation between system components that it expresses—an order of correlation unexplainable within classical physics.

Quantum dissonance is another form of quantum correlation, different from entanglement. Dissonance, so named by Modi *et al* [11] to refer to quantum discord [12] in separable quantum states, is comparatively less well understood, and it is important to discover when and to what degree it, too, can be a resource for quantum information processing. Dissonance is known to enable a statistical form of teleportation [13], it assists optimal state discrimination [14], it seems to be the driving resource for the DQC1 computational model [15], and there is evidence that it aids phase estimation [16]. Ideally, to show that dissonance is a quantum resource with standing similar to entanglement, one would show for the quantum protocol of interest that increasing dissonance consistently increases the protocol's performance. Just such demonstrations are available for entanglement for a variety of protocols [17]. Perhaps the strongest evidence yet for dissonance as a quantum resource is [14], in which cases are found where dissonance is identified to assist unambiguous state discrimination. Even that study, though, does not address the stringent question of whether *increasing* dissonance necessarily yields *increased* probability of discrimination. We show here for dissonant Bell-diagonal probes of the qubit depolarizing channel, that increased dissonance does necessarily yield greater probe information. This is the first instance in which quantum dissonance is demonstrated to act in this sense as a quantum resource for a standard quantum protocol.

Quantum channel identification [18], also known as quantum process tomography [1], is a standard quantum protocol for estimating channel parameters. Quantum channels are the fundamental building blocks of quantum information processing. In any physical implementation, though, the channel is usually not fully known and must be determined experimentally by observing its effect on prepared quantum systems, or probes. We study the qubit depolarizing channel because entanglement is known to aid the identification of this channel [18, 19]. Additionally, the depolarizing channel is a standard model of quantum noise, it has been the basis of investigation in a range of contexts [19–23], and it is analytically tractable for our purpose. Specifically, we study qubit depolarizing channel probes correlated with a varying degree of dissonance to a second ancilla qubit in a two-qubit Bell-diagonal state and use quantum Fisher information to gauge the amount of statistical information available from the probe about the channel depolarizing probability. The rich diversity of the Bell-diagonal states allows us to flexibly vary the dissonance between probe and ancilla while controlling other state features, including classical correlation and joint and marginal purities. A key point of this work is that the effect of dissonance on probe performance is only unambiguously revealed when these other factors are controlled for. Quantum Fisher information is the standard measure of probe performance in channel identification, and we prove that for estimating the quantum channel depolarizing probability, within classes of Bell-diagonal probe-ancilla states with constant classical correlation and constant purities, any increase or decrease in dissonance correspondingly increases or decreases the quantum Fisher information.

In the remainder of this paper's first section, we briefly review the channel identification problem and quantum Fisher information's role therein as well as quantum discord and dissonance. In section 2 we introduce parametric families of Bell-diagonal states for use as probe states, and we obtain simple expressions for probe state dissonance and quantum Fisher information within these families. Using these results, we see, in some typical examples, the need for controlling ancillary factors and the consequent gain in quantum Fisher information with increasing dissonance. In section 3 we show generally for the qubit depolarizing channel and Bell-diagonal probe states that the quantum Fisher information in the channel output is an increasing function of the probe state dissonance. We make some final, general remarks in section 4.

Channel identification and quantum Fisher information. The task of quantum channel identification is statistical by nature and is typically formulated as a parameter estimation problem: the unknown channel Γ_p is given to belong to a parametric family $\{\Gamma_p,\ p \in \mathcal{P}\}$ of channels indexed by the set $\mathcal{P}$, and we identify the channel by estimating p within $\mathcal{P}$. The scheme for doing this is to prepare the channel input probe in a chosen quantum state σ, make a quantum measurement of the channel output $\rho_p = \Gamma_p(\sigma)$, and record the measurement's registered result X. This process is repeated to obtain n independent, identically distributed measurement outcomes $X_1, \ldots, X_n$, and then p is estimated by an estimator $\hat{p} = \hat{p}(X_1, \ldots, X_n)$.

The quantum Fisher information $J(p) = J[\rho_p]$ in the parametric output $\rho_p = \Gamma_p(\sigma)$ of a quantum channel bounds the ultimate precision of the estimation of the parameter p attainable by quantum measurement of ρ_p. The quantum Cramér–Rao inequality states for n independent, identically distributed registrations of any quantum measurement and any unbiased estimator $\hat{p}$ that [24–26]

$$\mathrm{V}[\hat{p}] \geqslant \frac{1}{nJ(p)} \tag{1}$$

where $\mathrm{V}[\hat{p}]$ is the variance of $\hat{p}$. The quantum Fisher information $J(p)$ in (1) is

$$J(p) = J[\rho_p] = \mathrm{tr}[\rho_p L_p^2]$$

where L_p is the quantum score operator (symmetric logarithmic derivative) associated with ρ_p defined by

$$\frac{L_p\rho_p + \rho_p L_p}{2} = \partial_p \rho_p \tag{2}$$

where ∂_p signifies differentiation. The Cramér–Rao lower bound (1) is asymptotically $(n \to \infty)$ achievable so the larger $J(p)$ is in (1), the more precisely p can be estimated. We therefore interpret $J(p)$ to be the statistical information about p available in a parametrically defined quantum state ρ_p. For fixed p, $J(p)$ is a quantitative measure of the relative merit of different channel probe states.

The qubit depolarizing channel is a quantum channel [1] defined for any qubit input state σ by

$$\Gamma_p(\sigma) = \frac{1-p}{2}\mathbf{I} + p\sigma \tag{3}$$

where $1 - p$ is the probability of depolarization and $\frac{1}{2}\mathbf{I}$ is the completely mixed qubit state. Suppose the probe qubit is prepared in a two-qubit state ω with a second, ancilla qubit, and the probe qubit is passed through the channel while the ancilla qubit is held to the side. The channel output in this case is $(\Gamma_p \otimes \mathcal{I})(\omega)$ where $\mathcal{I}$ is the qubit identity channel. The quantum Fisher information $J[(\Gamma_p \otimes \mathcal{I})(\omega)]$ associated with the depolarizing channel has no simple expression in general, though simple expressions are known [19] for special input states ω. In the next section we derive a new expression for $J[(\Gamma_p \otimes \mathcal{I})(\omega)]$ generally applicable to two-qubit Bell-diagonal probe states.

Total correlation, discord and dissonance. The total correlation, quantum and classical, in subsystems A and B of a bipartite system in quantum state ω is quantified by the quantum mutual information, given by

$$I(\omega) = S(\omega_A) + S(\omega_B) - S(\omega) \tag{4}$$

where $\omega_A = \mathrm{tr}_B[\omega]$ and $\omega_B = \mathrm{tr}_A[\omega]$ are the marginal states of the two subsystems and $S(\rho) = -\mathrm{tr}[\rho \ln \rho]$ is the von Neumann entropy of the quantum state ρ [1]. The classical part of the total correlation (4) in subsystems A and B is related to the reduction in our uncertainty about the state of, say, A by measurement of B [12, 27]. Suppose we measure B by a quantum measurement $\{\mathbf{M}_k\}$ of von Neumann type with one-dimensional projectors $\mathbf{M}_k$ such that $\sum_k \mathbf{M}_k = \mathbf{I}$. This measurement casts the bipartite system, originally in state ω, into the state

$$\omega_k = \frac{1}{p_k}(\mathbf{I}\otimes\mathbf{M}_k)\omega(\mathbf{I}\otimes\mathbf{M}_k)$$

with probability $p_k = \mathrm{Tr}[(\mathbf{I}\otimes\mathbf{M}_k)\omega(\mathbf{I}\otimes\mathbf{M}_k)]$. Depending on the measurement outcome, the reduction in uncertainty about the state of A is $S(\omega_A) - S(\omega_k)$ with average reduction $S(\omega_A) - \sum_k p_k S(\omega_k)$. The supremum of this average reduction through measuring B is defined to be the classical part

$$C_A(\omega) = \sup_{\{\mathbf{M}_k\}}\left(S(\omega_A) - \sum_k p_k S(\omega_k)\right) \tag{5}$$

of the total correlation in ω. The optimization in (5) is more generally taken over quantum measurements described by positive operator-valued measures, but for two-qubit states the optimal measurement is known to be projective [28]. Definition (5), involving as it does measurement of subsystem B of the bipartite system, is not symmetrical in A and B and, in fact, $C_A(\omega) \neq C_B(\omega)$ generally [29]. This is not an issue for us because of the qubit exchange symmetry of the Bell-diagonal states that we consider, and we henceforth just write $C(\omega)$ for the classical correlation in a bipartite state ω. Alternative, symmetric definitions of classical correlation exist [29, 30] that yield the same result as (5) for Bell-diagonal states.

The quantum discord $D(\omega) = I(\omega) - C(\omega)$ in a bipartite state ω is the quantum part of the total correlation between the two qubits [12]. When ω is separable, any nonzero discord is due strictly to quantum correlation other than entanglement. Following Modi *et al* [11], we call positive discord in the absence of entanglement dissonance and say that a separable state with positive discord is dissonant.

2. Bell-diagonal probe states

Recent investigations [31, 32] involving quantum discord have been based on the Bell-diagonal states first studied in [33]. Because of their simple geometry [34], these states are a natural choice for the present study. The Bell-diagonal states are the two-qubit states

$$\varpi = \frac{1}{4}\left(\mathbf{I} + \sum_{j=1}^{3} c_j\sigma_j\otimes\sigma_j\right) \tag{6}$$

where $\mathbf{I}$ is the identity operator, σ_1, σ_2 and σ_3 are the Pauli operators and the state parameters c_j are real numbers. These states (6) have the eigendecomposition

$$\varpi = \lambda_0\Psi^- + \lambda_1\Phi^- + \lambda_2\Phi^+ + \lambda_3\Psi^+ , \tag{7}$$

with eigenvalues

$$\lambda_0 = \frac{1-c_1-c_2-c_3}{4}$$
$$\lambda_1 = \frac{1-c_1+c_2+c_3}{4}$$
$$\lambda_2 = \frac{1+c_1-c_2+c_3}{4}$$
$$\lambda_3 = \frac{1+c_1+c_2-c_3}{4}$$

and Bell eigenstates

$$\Phi^{\pm} = |\phi^{\pm}\rangle\langle\phi^{\pm}| \,, \quad \Psi^{\pm} = |\psi^{\pm}\rangle\langle\psi^{\pm}| \tag{8}$$

where

$$|\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \,, \quad |\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \,.$$

The conditions $\lambda_j \geqslant 0$ on the eigenvalues of ϖ constrain its triple (c_1, c_2, c_3) of state parameters to lie in the tetrahedron $\mathcal{T}$ with vertices $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(-1, -1, -1)$. When $(c_1, c_2, c_3) \in \mathcal{T}$ we write also $\varpi \in \mathcal{T}$ in a minor abuse of notation.

The Bell-diagonal states $\varpi \in \mathcal{T}$ have a number of simple, well-known properties. For all $\varpi \in \mathcal{T}$ the marginal states of qubits A and B are completely mixed and, in fact, every two-qubit state with completely mixed marginals is locally unitarily equivalent to some Bell-diagonal state [35]. The state $\varpi \in \mathcal{T}$ is readily found by the PPT criterion [36] to be separable if and only if (c_1, c_2, c_3) lies within the octahedron $\mathcal{O} \subset \mathcal{T}$ defined by $|c_1| + |c_2| + |c_3| \leqslant 1$ (see figure in [33] or [34]).

The joint purity of a qubit pair in a state $\varpi \in \mathcal{T}$ varies with ϖ even though the purities of the individual qubits do not. The joint purity

$$\mathrm{tr}\, \varpi^2 = \frac{1 + c_1^2 + c_2^2 + c_3^2}{4} = \frac{1 + s^2}{4} \tag{9}$$

depends specifically on just the 'radius' $s = \sqrt{c_1^2 + c_2^2 + c_3^2}$ of the state. According to (9), all states $\varpi \in \mathcal{O}$ are mixed, with constant purity on any given centered sphere

$$\mathcal{S}(s) = \left\{(c_1, c_2, c_3) : c_1^2 + c_2^2 + c_3^2 = s^2\right\}. \tag{10}$$

The largest such sphere in $\mathcal{O}$ is $\mathcal{S}(1/\sqrt{3})$; among such spheres these states $\varpi \in \mathcal{S}(1/\sqrt{3})$ are the purest states, with purity $\mathrm{tr}\, \varpi^2 = 1/3$ (in the possible range $[1/4, 1]$).

The total correlation (4) in two qubits A and B in a Bell-diagonal state $\varpi \in \mathcal{T}$ in the form (7) is

$$I(\varpi) = 2 + \sum_{j=0}^{3} \lambda_j \log_2 \lambda_j \tag{11}$$

because the qubits are individually completely mixed, for which case $S(\omega_A) = S(\omega_B) = 1$. The classical part $C(\varpi)$ of this total correlation is [35]

$$C(\varpi) = 1 - h\left(\frac{1 + c}{2}, \frac{1 - c}{2}\right) \tag{12}$$

where $c = \max(|c_1|, |c_2|, |c_3|)$ and

$$h(x_1, x_2) = x_1 \log_2 \frac{1}{x_1} + x_2 \log_2 \frac{1}{x_2} \,. \tag{13}$$

The classical correlation $C(\varpi)$ is constant for fixed $c = \max(|c_1|, |c_2|, |c_3|)$. Thus $C(\varpi)$ is constant on the surface $\mathcal{C}(c)$ of the cube centered at the origin, oriented with edges parallel to the c_1, c_2, c_3 axes, and with edge half-length c. Expressions (11) and (12) allow us to readily calculate the discord $D(\varpi) = I(\varpi) - C(\varpi)$ for any $\varpi \in \mathcal{T}$. For any separable Bell-diagonal state $\varpi \in \mathcal{O}$, this is the state dissonance.

Suppose we prepare a probe qubit in a Bell-diagonal state $\varpi \in \mathcal{T}$ with an ancilla qubit and pass the probe qubit through the depolarizing channel (3). The channel output is readily shown from the special form (7) of ϖ to be

$$(\Gamma \otimes \mathcal{I})(\varpi) = p\varpi + \frac{1 - p}{4}\mathbf{I} \tag{14}$$

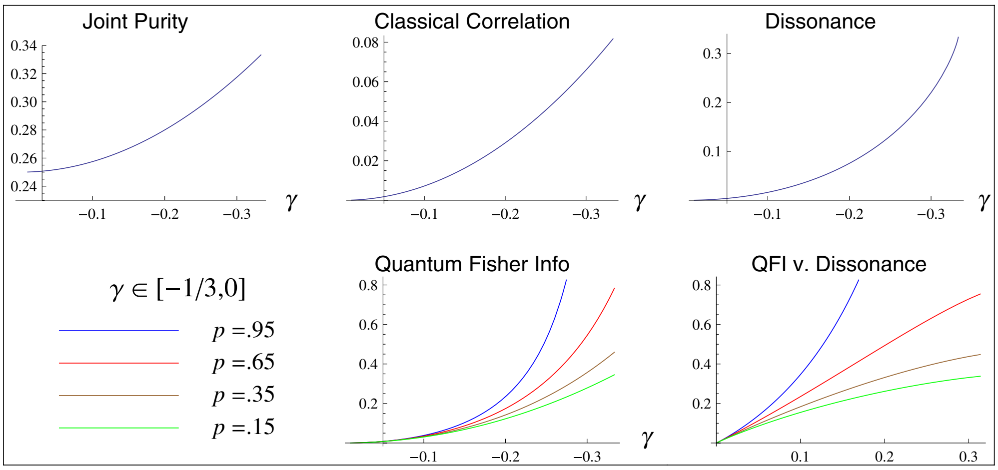

Figure 1. Joint purity, classical correlation and dissonance of the probe state and the quantum Fisher information in the channel output as the probe state is varied through the negative Werner states, $\gamma \in [-1/3, 0]$.

with eigenvalues

$$w_j(p) = p\lambda_j + \frac{1-p}{4}$$

and associated derivatives $w_j'(p) = \lambda_j - 1/4$. The channel output state (14) is a convex combination of ϖ and the two-qubit completely mixed state $\frac{1}{4}\mathbf{I}$. These two states commute, sharing a common eigenbasis composed of the Bell states (8). For an output state of this convex form, the score operator L_p in (2) has a fixed eigenbasis (the Bell states) independent of p, and the quantum Fisher information is [19, 26]

$$J(p) = \sum_{j=0}^{3} \frac{(w_j'(p))^2}{w_j(p)} = \sum_{j=0}^{3} \frac{(\lambda_j - 1/4)^2}{p\lambda_j + (1-p)/4}\ . \tag{15}$$

Some further calculation (using $\sum_j \lambda_j = 1$) yields the result that, for any Bell-diagonal probe state $\varpi \in \mathcal{T}$, the quantum Fisher information about p in the depolarizing channel output is

$$J(p) = \frac{1}{p(1-p)} - \frac{1}{p(1-p)} \sum_{j=0}^{3} \frac{\lambda_j}{1 - p + 4p\lambda_j}\ . \tag{16}$$

With expressions (11), (12) and (16) we can now proceed to relate probe dissonance to quantum Fisher information for different $\varpi \in \mathcal{O}$. A natural one-parameter family of Bell-diagonal states for this purpose is the 'negative' Werner states. The standard Werner states are the two-qubit states

$$\rho = \frac{1-\gamma}{4}\mathbf{I} + \gamma\Psi^- \tag{17}$$

with $\gamma \in [0, 1]$. Negative Werner states have the form (17), but with $\gamma \in [-1/3, 0]$. The negative Werner states are Bell-diagonal states, they have zero entanglement for all $\gamma \in [-1/3, 0]$, and among separable Bell-diagonal states they vary monotonically from $D(\rho) = 0$ at $\gamma = 0$ to $D(\rho) = 1/3$ at $\gamma = -1/3$ (see figure 1), exhibiting the widest possible range of dissonance[3]. The dissonance of the negative Werner states is shown in figure 1,

[3] The most dissonant Bell-diagonal state corresponds to the unique point $(c_1, c_2, c_3) = (-1/3, -1/3, -1/3) \in \mathcal{O}$ (disregarding trivial symmetries) [34, 37]. This is the negative Werner state with $\gamma = -1/3$.

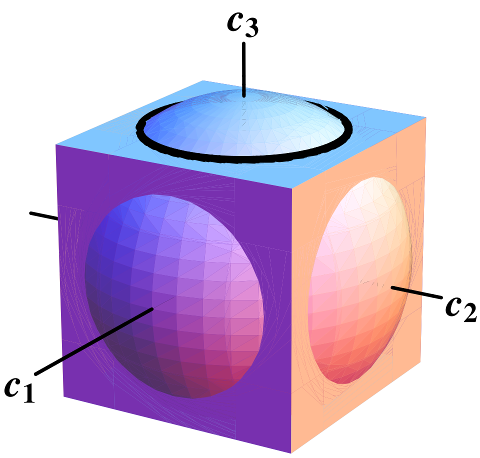

Figure 2. The sphere has radius $s = .3$. The cube has edge half-length $c = .38$. The solid circle is the circle family $\mathcal{L}(.3, .38)$ of separable Bell-diagonal states, with constant purity and constant classical correlation.

along with the quantum Fisher information (16) when these are used as probe states for the depolarizing channel. We see stark increases in the quantum Fisher information as the probe dissonance increases. The increase in dissonance across the negative Werner states, however, is coupled with increases in both state purity and classical correlation, and we cannot conclude that increased probe dissonance is the cause of the information increase. In fact, state purity is known to affect the quantum information [18], and it may be part or all of the source of the increase in information seen in figure 1. The concurrent variation of joint purity, classical correlation, dissonance and quantum Fisher information seen in the negative Werner states is typical of parametric families of separable Bell-diagonal states. Families of two-qubit states wherein only the dissonance varies are needed. Recalling that the separable Bell-diagonal states have constant purity on $\mathcal{S}(s)$ and constant classical correlation on $\mathcal{C}(c)$, we introduce for this purpose the circle families $\mathcal{L}(c, s) = \mathcal{C}(c) \cap \mathcal{S}(s)$ of separable Bell-diagonal states.

The circle family $\mathcal{L}(c, s)$ for given $s > c > 0$ is the one-parameter family of separable Bell-diagonal states $\varpi \in \mathcal{O}$ with state parameters given for $\theta \in [0, 2\pi]$ by

$$c_1 = r\cos\theta, \quad c_2 = r\sin\theta, \quad c_3 = c \tag{18}$$

where $r = \sqrt{s^2 - c^2}$ is the radius of the circle family $\mathcal{L}(c, s)$. The states $\varpi \in \mathcal{L}(c, s)$ occupy in (c_1, c_2, c_3) parameter space the intersection of $\mathcal{S}(s)$ and $\mathcal{C}(c)$. This intersection is a circle within $\mathcal{O}$ for suitably chosen c and s. (In fact, $\mathcal{S}(s)$ and $\mathcal{C}(c)$ intersect in six circles, but the other five circles offer no additional physics, and we give all our attention to (18).) The family $\mathcal{L}(c, s)$ is a circle within the octahedron $\mathcal{O}$ only if $0 < c < 1$. For small $c > 0$, we only need $c < s \leqslant \sqrt{2}c$ for a circular intersection within $\mathcal{O}$. For larger $c < 1$ the octahedron $\mathcal{O}$ constrains s to $s \leqslant \sqrt{c^2 + \frac{1}{2}(1-c)^2}$. Therefore, for $0 < c < 1$ and $c < s \leqslant \min(\sqrt{2}c, \sqrt{c^2 + \frac{1}{2}(1-c)^2})$, $\mathcal{S}(s)$ and $\mathcal{C}(c)$ intersect in a circle within $\mathcal{O}$, and we call $\mathcal{L}(c, s) = \mathcal{S}(s) \cap \mathcal{C}(c) \subset \mathcal{O}$ a circle family of Bell-diagonal states; the circle family $\mathcal{L}(.3, .38)$ is shown in figure 2. The region of admissible pairs (c, s) for circle families $\mathcal{L}(c, s)$ is shown in figure 3.

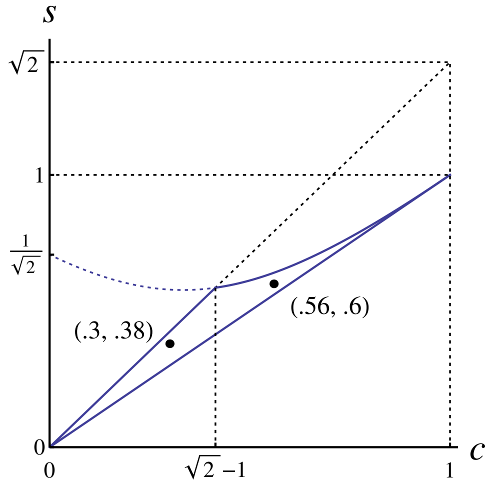

Figure 3. The set $\{0 < c < 1, c < s \leqslant \min(\sqrt{2}c, \sqrt{c^2 + \frac{1}{2}(1-c)^2})\}$ of admissible half-edge/radius pairs (c, s) for circle families $\mathcal{L}(c, s)$. Two typical pairs explored in the text are identified.

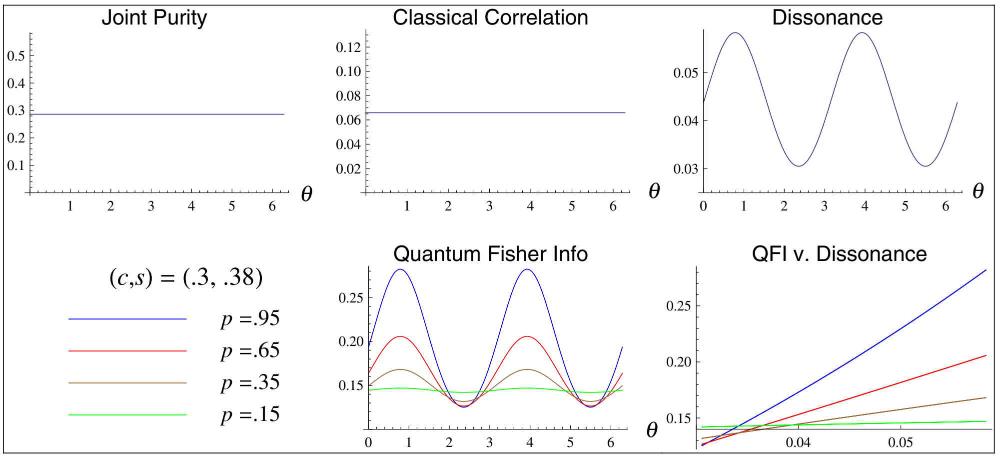

Figure 4. Joint purity, classical correlation and dissonance of the probe state and the quantum Fisher information in the channel output as the probe state is varied around $\mathcal{L}(.3, .38)$.

The circle families of Bell-diagonal states are useful because, within any given circle family $\mathcal{L}(c, s)$, the joint purity and classical correlation between the probe and its ancilla are constant. Also, of course, $\mathcal{L}(c, s) \subset \mathcal{O}$ so the entanglement and marginal purities are fixed—to zero entanglement and completely mixed marginals. The only readily identifiable property that varies as we move through the states of $\mathcal{L}(c, s)$ is the dissonance of the probe state. This allows us to look at the effect of dissonance on probe information in the absence of other factors. Results for two typical circle families $\mathcal{L}(.3, .38)$ and $\mathcal{L}(.56, .6)$ (see figure 3) are shown in figures 4 and 5. We see unambiguously in these two examples that the probe dissonance is acting as an aid for channel probing. We show in the next section that this effect is present in any circle family of Bell-diagonal states.

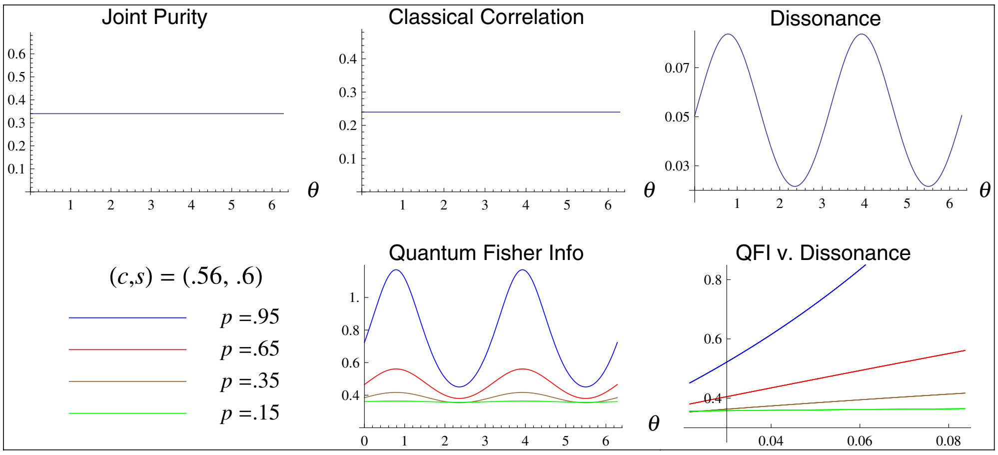

Figure 5. Joint purity, classical correlation and dissonance of the probe state and the quantum Fisher information in the channel output as the probe state is varied around $\mathcal{L}(.56, .6)$.

3. $\mathcal{L}$(c, s) probe states

We saw in the previous section in the cases of two particular circle families $\mathcal{L}(c, s)$ of separable Bell-diagonal probe states that the quantum Fisher information in the probe state at the channel output consistently increased with any increase in the probe's input state dissonance. Significantly, in these two cases both the joint purity and the classical correlation of the probe and its ancilla were held constant. We now prove that when the dissonance is varied within any circle family $\mathcal{L}(c, s)$, the quantum Fisher information always increases with greater dissonance. Because all other pertinent features, including purity, classical correlation and entanglement, are fixed in this setting, this indicates unambiguously that dissonance is acting here as an aid to channel identification.

Suppose that to probe the depolarizing channel (3) we prepare probe and ancilla qubits in a joint Bell-diagonal state $\varpi \in \mathcal{L}(c, s) \subset \mathcal{O}$ with state parameters c_j defined as in (18) for some admissible (c, s). For convenience let $\phi = \theta + \pi/4$. Then the eigenvalues of ϖ are

$$\lambda_0 = \frac{1 - c - s\sqrt{2}\sin\phi}{4}, \qquad \lambda_1 = \frac{1 + c - s\sqrt{2}\cos\phi}{4},$$
$$\lambda_2 = \frac{1 + c + s\sqrt{2}\cos\phi}{4}, \qquad \lambda_3 = \frac{1 - c + s\sqrt{2}\sin\phi}{4}$$

where $r = \sqrt{s^2 - c^2}$. Using the general result (16) obtained for the quantum Fisher information $J(p)$ for any probe state $\varpi \in \mathcal{T}$, we find that the change in $J(p)$ as we advance ϕ to move the probe state ϖ around the circle $\mathcal{L}(c, s)$ is

$$\begin{aligned}\frac{\partial J(p)}{\partial\phi} &= -\frac{1}{p}\sum_{j=0}^{3}\frac{\lambda_j'}{\left(1 - p + 4p\lambda_j\right)^2}\\ &= -\frac{1}{p}\frac{\lambda_0'}{\left(1 - pc + 4p\lambda_2'\right)^2} - \frac{1}{p}\frac{\lambda_1'}{\left(1 + pc + 4p\lambda_0'\right)^2}\\ &\quad - \frac{1}{p}\frac{\lambda_2'}{\left(1 + pc + 4p\lambda_3'\right)^2} - \frac{1}{p}\frac{\lambda_3'}{\left(1 - pc + 4p\lambda_1'\right)^2}\end{aligned} \tag{19}$$

where the λ_j' are the derivatives

$$\lambda_0' = -\frac{\sqrt{2}r}{4}\cos\phi\ , \quad \lambda_1' = \frac{\sqrt{2}r}{4}\sin\phi\ ,$$
$$\lambda_2' = -\frac{\sqrt{2}r}{4}\sin\phi\ , \quad \lambda_3' = \frac{\sqrt{2}r}{4}\cos\phi\ .$$

Using the relations $\lambda_0' = -\lambda_3'$ and $\lambda_2' = -\lambda_1'$, we have from (19) that

$$\begin{aligned}\frac{\partial J(p)}{\partial\phi} &= \frac{\lambda_3'}{p}\left(\frac{1}{\left(1-pc-4p\lambda_1'\right)^2} - \frac{1}{\left(1-pc+4p\lambda_1'\right)^2}\right) \\ &\quad - \frac{\lambda_1'}{p}\left(\frac{1}{\left(1+pc-4p\lambda_3'\right)^2} - \frac{1}{\left(1+pc+4p\lambda_3'\right)^2}\right) \\ &= \frac{\lambda_3'}{p}\frac{16p(1-pc)\lambda_1'}{\left((1-pc)^2-16p^2{\lambda_1'}^2\right)^2} - \frac{\lambda_1'}{p}\frac{16p(1+pc)\lambda_3'}{\left((1+pc)^2-16p^2{\lambda_3'}^2\right)^2} \\ &= 16\lambda_1'\lambda_3'\left(\frac{1-pc}{\left((1-pc)^2-16p^2{\lambda_1'}^2\right)^2} - \frac{1+pc}{\left((1+pc)^2-16p^2{\lambda_3'}^2\right)^2}\right).\end{aligned} \tag{20}$$

Writing (20) with a common denominator, we have finally

$$\frac{\partial J(p)}{\partial\phi} = r^2 A(p,c,s,\phi)\sin 2\phi \tag{21}$$

where

$$A(p,c,r,\phi) = \frac{(1-pc)Y^2-(1+pc)X^2}{X^2Y^2} \tag{22}$$

with

$$X = (1-pc)^2 - 2p^2r^2\sin^2\phi\ ,$$
$$Y = (1+pc)^2 - 2p^2r^2\cos^2\phi\ .$$

We are primarily interested in conditions for which $A(p,c,r,\phi)$ is numerically positive. Considering $\phi = 0$ and noting that $r^2 \leqslant c^2$, we readily find that $Y \geqslant 1+pc$ and $X \leqslant 1-pc$ for $p > 0$. Then $Y^2 \geqslant 1+pc$ and $X^2 \leqslant 1-pc$, and it is straightforward to show that the numerator of $A(p,c,r,\phi)$ in (22) is positive for $p > 0$ for any state in any admissible circle family $\mathcal{L}(c,s)$.

Now consider the change in the quantum dissonance as we move the probe state ϖ around the circle $\mathcal{L}(c,s)$ by advancing ϕ. We have

$$\begin{aligned}\frac{\partial D(\varpi)}{\partial\phi} &= \frac{\partial I(\varpi)}{\partial\phi} - \frac{\partial C(\varpi)}{\partial\phi} \\ &= \frac{\partial I(\varpi)}{\partial\phi} \\ &= \sum_{j=0}^{3}\lambda_j'\log_2\lambda_j\end{aligned}$$

$$= \frac{\sqrt{2}s}{4\ln 2}\left[\cos\phi \ln\frac{\lambda_3}{\lambda_0} - \sin\phi\ln\frac{\lambda_2}{\lambda_1}\right]$$
$$= \frac{s}{\sqrt{2}\ln 2}\left[\cos\phi\tanh^{-1}\frac{r\sqrt{2}\sin\phi}{1-c} - \sin\phi\tanh^{-1}\frac{r\sqrt{2}\cos\phi}{1+c}\right]. \tag{23}$$

The arguments of $\tanh^{-1}$ in (23) are less than 1 in magnitude for states belonging to any admissible circle family. We therefore expand the $\tanh^{-1}$ functions in (23) in their MacLaurin series

$$\tanh^{-1}x = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}, \quad |x| < 1$$

and find

$$\frac{\partial D(\varpi)}{\partial\phi} = r^2 B(c, r, \phi)\sin 2\phi \tag{24}$$

where

$$B(c, r, \phi) = \frac{1}{2\ln 2}\left[\frac{1}{1-c}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{r\sqrt{2}\sin\phi}{1-c}\right)^{2k} - \frac{1}{1+c}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{r\sqrt{2}\cos\phi}{1+c}\right)^{2k}\right]$$

with $r = \sqrt{s^2 - c^2}$. We have

$$B(c, r, \phi) \geqslant B(c, r, 0)$$
$$= \frac{1}{2\ln 2}\left[\frac{1}{1-c} - \frac{1}{1+c}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{r\sqrt{2}}{1+c}\right)^{2k}\right]. \tag{25}$$

This new lower bound (25) is a decreasing function of r, and $r \leqslant c$ so

$$B(c, r, \phi) \geqslant B(c, c, 0)$$
$$= \frac{1}{2\ln 2}\left[\frac{1}{1-c} - \frac{1}{1+c}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{c\sqrt{2}}{1+c}\right)^{2k}\right]$$
$$= \frac{1}{2\ln 2}\left[\frac{1}{1-c} - \frac{1}{c\sqrt{2}}\tanh^{-1}\frac{c\sqrt{2}}{1+c}\right],$$

and in this last form $B(c, c, 0)$ is readily seen to be positive for all admissible $0 < c < 1$. We now combine expressions (21) and (24) to obtain

$$\frac{\partial J(p)}{\partial D(\varpi)} = \frac{A(p, c, r, \phi)}{B(c, r, \phi)} > 0 \tag{26}$$

for the change in the quantum Fisher information relative to the change in dissonance as we vary ϕ to move through the circle family $\mathcal{L}(c, s)$ of Bell-diagonal probe states. We see from (26) that the quantum Fisher information is strictly increasing with the probe dissonance and that this is true for any $p > 0$ and any circle family $\mathcal{L}(c, s)$. This is what we set out to prove in this section.

Our main result (26) is evidence that, for probing the qubit depolarizing channel by a separable Bell-diagonal probe state, dissonance is advantageous. This does not mean, though, that a state belonging to $\mathcal{O}$ with positive dissonance necessarily yields more information than any zero-dissonance state in $\mathcal{O}$. In fact $\mathcal{O}$ contains states with zero dissonance that out-perform states in $\mathcal{O}$ with positive dissonance. Consider the state $\tau \in \mathcal{O}$ with $(c_1, c_2, c_3) = (1/\sqrt{3}, 0, 0)$.

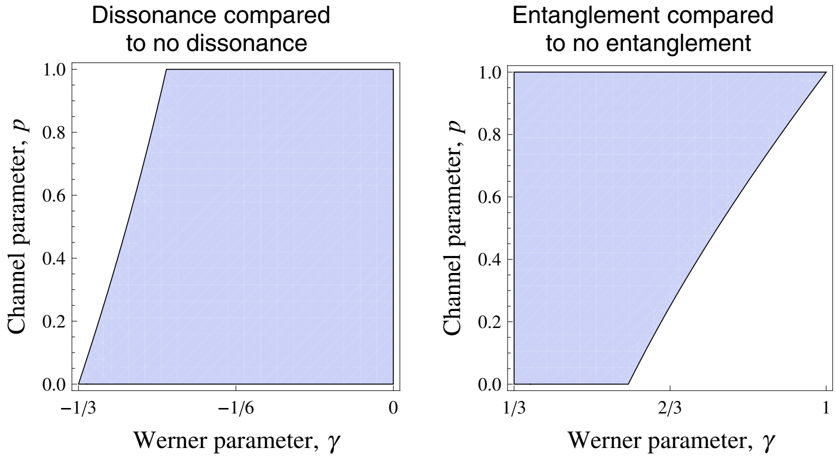

Figure 6. The shaded region in the left panel is the pairs (γ, p) of negative Werner state and depolarizing channel parameter values where the zero-dissonance probe state τ (see text) yields more information than does a dissonant probe state. The shaded region in the right panel is the pairs (γ, p) of (standard) Werner state and depolarizing channel parameter values where the zero-entanglement probe state $|\psi\rangle$ (see text) yields more information than does an entangled probe.

This state has dissonance $D(\tau) = 0$. Compare τ with a state ρ from among the negative Werner states (17). These states have dissonance $D(\rho) > 0$ for $\gamma \in (-1/3, 0)$. The probe states τ and ρ yield, according to (16), the respective quantum Fisher informations

$$J(p;\tau) = \frac{1}{3-p^2}\,, \qquad J(p;\rho) = \frac{3\gamma^2}{(1-p\gamma)(1+3p\gamma)}\,, \qquad \gamma \in (-1/3, 0)\,. \tag{27}$$

As shown in figure 6 (left panel), we find for $\gamma \geqslant (\sqrt{10}-1)/9$ that $J(p;\tau) > J(p;\rho)$ and for $\gamma < (\sqrt{10}-1)/9$ that sometimes $J(p;\tau) < J(p;\rho)$ and sometimes $J(p;\tau) > J(p;\rho)$. Thus here are many examples of probe states ρ with positive dissonance that yield less information than the probe state τ with zero dissonance. This, however, neither contradicts (26) nor refutes our conclusion therefrom that dissonance appears to be a quantum resource for probing the depolarizing channel. This is because the comparative effects of τ and ρ are confounded by the states' differing purities and classical correlations. In fact, τ is both purer and has more classical correlation ($\mathrm{tr}\rho^2 < \mathrm{tr}\,\tau^2 = 1/3$ and $C(\rho) < C(\tau) = 1/\sqrt{3}$) than any of the negative Werner states ρ with $\gamma \in (-1/3, 0)$. Thus, even though τ has zero dissonance, it often yields more information than ρ simply because, in such cases, it has sufficiently greater purity and classical correlation. To be meaningful, comparisons of the effects of dissonance must be made among quantum states that are otherwise the same.

The same holds true in investigations of entanglement; comparisons of the effects of entanglement must be made among states that are otherwise the same. Otherwise, unentangled probe states are readily found that out-perform entangled states. Suppose, for example, we consider probing the qubit depolarizing channel with states with varying degrees of entanglement. Among pure states, any increase in entanglement is accompanied by an increase in quantum Fisher information [19]. By restricting our attention to just pure states, we are able to conclude that entanglement has, in this setting, the nature of a quantum resource. But suppose we were to compare a pure unentangled probe state $|\psi\rangle$ and a standard Werner state

ρ in (17) with $\gamma \in (1/3, 1)$. For $\gamma \in (1/3, 1)$ the state ρ is entangled. According to (16), the quantum Fisher informations associated with $|\psi\rangle$ and ρ are

$$J(p; |\psi\rangle) = \frac{1}{1-p^2}, \qquad J(p; \rho) = \frac{3\gamma^2}{(1-p\gamma)(1+3p\gamma)}, \qquad \gamma \in (1/3, 1). \tag{28}$$

As shown in figure 6 (right panel), we find for $\gamma \leqslant 1/\sqrt{3}$ that $J(p; |\psi\rangle) > J(p; \rho)$ and for $\gamma > 1/\sqrt{3}$ that sometimes $J(p; |\psi\rangle) < J(p; \rho)$ and sometimes $J(p; |\psi\rangle) > J(p; \rho)$. Many entangled states ρ in this comparison fail to yield more information than the unentangled state $|\psi\rangle$. They fail, not because entanglement is not a quantum resource, but because the comparison is confounded by other types of correlation and differing purities—$|\psi\rangle$ is a pure state with no correlations, classical or otherwise, while ρ is strictly mixed with both classical correlation and quantum correlation beyond entanglement [35]. It is impossible (with present theory) to isolate the role of entanglement in these comparisons. To definitively assess entanglement—or any state property—as a quantum resource, comparisons of its effects must be made among states that are otherwise the same. Studying dissonance in this manner as we have done in the present work, we find that dissonance has the nature of a quantum resource for probing the qubit depolarizing channel.

4. Final remarks

Dissonant quantum states are necessarily mixed states, typically highly mixed. In fact, the purest separable Bell-diagonal state has purity only $\mathrm{tr}\,\varpi^2 = \frac{1}{2}$. Since pure states are optimal for channel probing [18], unless one has a physical probing apparatus that only allows mixed state probes, it is not clear that dissonance can, as a practical matter, be an attractive resource for channel probing. In any case, our intent here is not to promote dissonance as a practical resource. Rather, we have presented new evidence that—within the present setting of the qubit depolarizing channel, Bell-diagonal probe states, etc—dissonance has the theoretical properties of a quantum information processing resource. This encourages us to ask about its properties as a resource for other tasks in quantum information processing, and it is likely that the circle families of Bell-diagonal states introduced here can be useful again in those investigations.

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