

Topic 1

Charge, Current, Voltage & Resistance; Ohm's "Law"

1.1 Charge

- SI unit is Coulombs (C).
- Fundamental unit is charge on electron:

$$e = 1.6 \times 10^{-19} \text{ C}$$

1.2 Current

- Current is a measure of the flow rate of charge past a point in a circuit:

$$I = \frac{Q}{t} \longrightarrow \frac{dQ}{dt}.$$

- Current can be written in terms of microscopic quantities as

$$I = nAve,$$

where n is the density of electrons, A is the cross-sectional area of the wire, v is the *drift velocity* of the electrons, and e is the charge of the electron.

- SI unit is Coulombs/second (C/s), or Amperes (A).
- Ammeters measure current that flows through the meter. Ideal ammeters should have no resistance.

1.3 Voltage

- Formal definition: The voltage *difference* between two points A and B in an electrical circuit is the negative of the work done *per unit charge* by electric fields in moving a charge from point A to point B.
- Ideal batteries are sources of constant voltage difference
Bench-top power supplies are sources of constant voltage difference. (The BK Precision supplies are sources of constant voltage difference when they are in constant voltage, or **CV**, mode.)
- A voltage difference between the ends of a circuit element causes electric fields within the elements that can result in current through the element.
- Ideal voltmeters have infinite resistance so that no current flows through them.
- Voltage is sometimes called electric potential; voltage difference is equivalent to potential difference.
- SI unit for voltage/potential is Joules/Coulomb (J/C), or Volts (V)

1.4 Resistance and Ohm's Law

In some materials there is a direct proportionality between the potential difference between two points in a circuit and the current that flows between the two points. Such materials are said to obey Ohm's Law. (This is not really a law, but rather an observation about some materials.) In other words,

$$\Delta V = IR \quad \text{or} \quad I = \frac{\Delta V}{R}.$$

1.5 Resistance, Resistivity, and Conductivity

The resistance of a wire can be written in terms of a material dependent quantity ρ , the resistivity, and geometric factors:

$$R = \rho \frac{L}{A},$$

where L is the length of the wire and A is the cross-sectional area. This same relationship can also be expressed in terms of the conductivity of the material, σ ,

which is just the inverse of the resistivity:

$$R = \frac{1}{\sigma} \frac{L}{A},$$

Reading

- Simpson, 1.1–1.5
- Horowitz & Hill, 1.01–1.02
- O'Malley, Chapters 1–2 (there is some additional material in these chapters)
- Nahvi & Edminster, Chapters 1–2 (there is some additional material in these chapters)
- Scherz, 2.1–2.5, 14.3–14.4 (meters and scopes)

Topic 2

DC Circuits and Kirchoff's Laws

2.1 DC Circuits

- Terminology: Node/Junction, Branch, Loop, Open Circuits/Closed Circuits, Circuit Elements in Series, Circuit Elements in Parallel
- For resistors in series:

$$R_{\text{eq}} = R_1 + R_2$$

- For resistors in parallel:

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}$$

2.2 Kirchoff's Current Law

Conservation of charge implies that the sum of the currents flowing into a junction equals the sum of the currents flowing out of the junction:

$$\sum_i I_i = 0$$

2.3 Kirchoff's Voltage Law

Conservation of energy implies that the sum of the voltage drops around any closed loop in a circuit must equal zero:

$$\sum_i \Delta V_i = 0$$

2.4 Power

- The power (work per unit time) dissipated in a circuit element is given by the (work/charge) \times (charge/time), or

$$P = I \Delta V.$$

- The SI unit of power is Joules/second (J/s) or Watts (W).
- For circuit elements that obey Ohm's Law,

$$P = I^2 R = \frac{(\Delta V)^2}{R}.$$

Reading

- Simpson, 1.6–1.8, 1.14 (Wheatstone Bridge)
- Horowitz & Hill, 1.01–1.02
- O'Malley, Chapter 3, p. 86 in Chapter 4 (Wheatstone Bridge)
- Nahvi & Edminster, Chapter 3
- Scherz, 2.11, 2.12 (through example 1 on p. 55), 2.17-18

Topic 3

Voltage Dividers and Equivalent Circuits

3.1 Voltage Dividers

Simple application of Kirchoff's Laws to resistors in series (or resistors that are effectively in series) with one end grounded:

$$V_{\text{out}} = \frac{R_1}{R_1 + R_2} V_{\text{in}}.$$

3.2 Voltage Dividers as Non-Ideal Voltage Sources

Handout exercise.

3.3 Equivalent Circuits

Thévenin's Theorem: Any two-terminal network of resistors and voltage sources is equivalent to a single resistor (R_{Th}) in series with a single ideal voltage source (V_{Th}).

$$V_{\text{Th}} = \text{Open circuit voltage}$$

$$R_{\text{Th}} = \frac{\text{Open circuit voltage}}{\text{Short circuit current}}$$

or

R_{Th} = Resistance between terminals with all voltage sources “shorted out”

3.4 Thévenin Equivalent of a Voltage Divider

For a voltage divider:

$$V_{\text{Th}} = \frac{R_1}{R_1 + R_2} V_{\text{in}}.$$

$$R_{\text{Th}} = R_{\text{parallel}} = \frac{R_1 R_2}{R_1 + R_2}$$

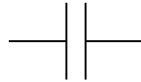
Reading

- Simpson, 1.9–1.12 (1.13 for Norton equivalent circuits)
- Horowitz & Hill, 1.03–1.05
- O’Malley, Chapter 5 through p. 86
- Nahvi & Edminster, Chapter 4.7–4.10
- Scherz, 2.13, 2.19

Topic 4

RC Circuits: Transient Response to Step Functions

4.1 Capacitors



- Capacitors are physical breaks in circuits.
- Capacitors store an amount of charge that is proportional to the potential difference across the capacitor:

$$Q_C = C \Delta V \quad \text{or} \quad C = \frac{Q_C}{\Delta V}.$$

- SI unit of capacitance is Farads (F).
- Capacitors block DC current; pass (in a sense) AC current.
- Capacitors in parallel:

$$C_{\text{eq}} = C_1 + C_2.$$

- Capacitors in series:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{or} \quad C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}.$$

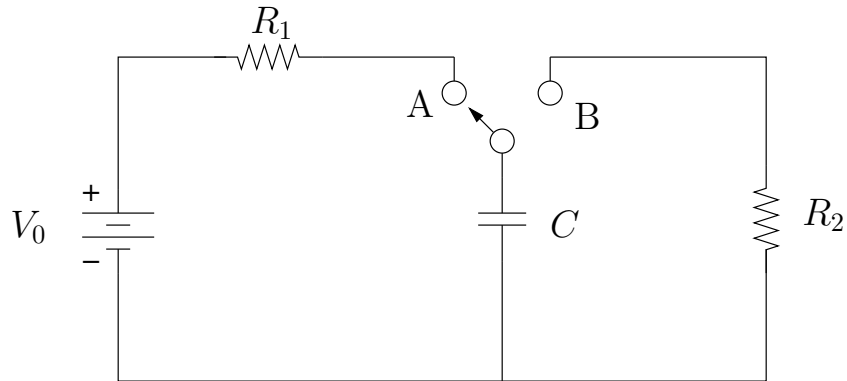
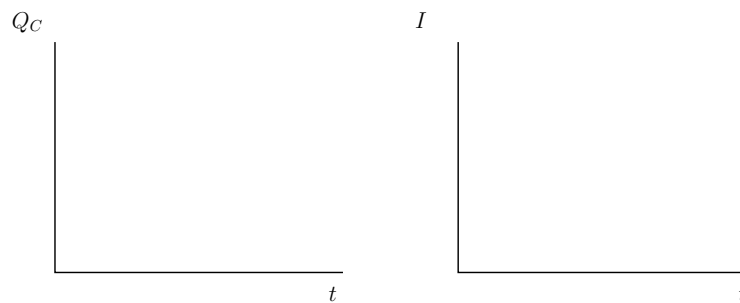


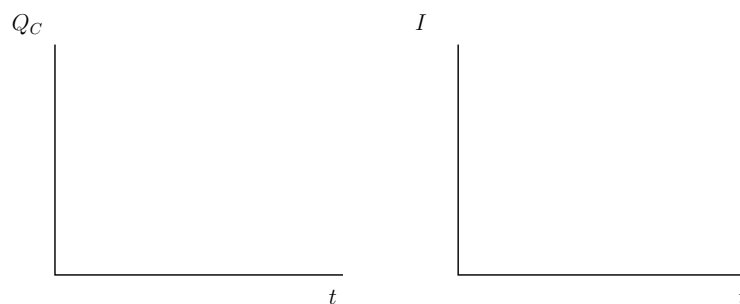
Figure 4.1: Simple RC circuit.

4.2 Charging and discharging in RC circuits (qualitative)

- Charging of a capacitor through a resistor:



- Discharging of a capacitor through a resistor:



4.3 Differential Equations

- The solution of a differential equation is a function.
- Functional solutions of differential equations have undetermined constants.
- Undetermined constants can be fixed by considering *boundary conditions*.

4.4 Charging and discharging in RC circuits (quantitative)

- Derivations: Kirchoff's loop rule \longrightarrow differential equation.
- Charging of a capacitor through a resistor
Kirchoff's Loop Rule:

$$V_0 - IR - \frac{Q_C}{C} = 0 \longrightarrow Q_C - RC \frac{dQ_C}{dt} = V_0 C$$

Initial Condition:

$$Q_C(0) = 0.$$

Solution:

$$Q_C(t) = CV_0 (1 - e^{-t/RC}) \longrightarrow V_C(t) = V_0 (1 - e^{-t/RC})$$

and

$$V_R(t) = I(t)R = \frac{dQ_C}{dt} R = V_0 e^{-t/RC}.$$

- Discharging of a capacitor through a resistor
Kirchoff's Loop Rule:

$$V_0 + IR = 0 \longrightarrow \frac{Q_C}{C} + \frac{dQ}{dt} R \longrightarrow \frac{dQ_C}{dt} = -\frac{1}{RC} Q_C$$

Initial Condition:

$$Q_C(0) = Q_0 = CV_0$$

Solution:

$$Q_C(t) = Q_0 e^{-t/RC} \longrightarrow V_C(t) = V_0 e^{-t/RC},$$

and

$$V_R(t) = I(t)R = \frac{dQ}{dt} R = -V_0 e^{-t/RC}.$$

- Characteristic time/decay time/time constant: $\tau = RC$.

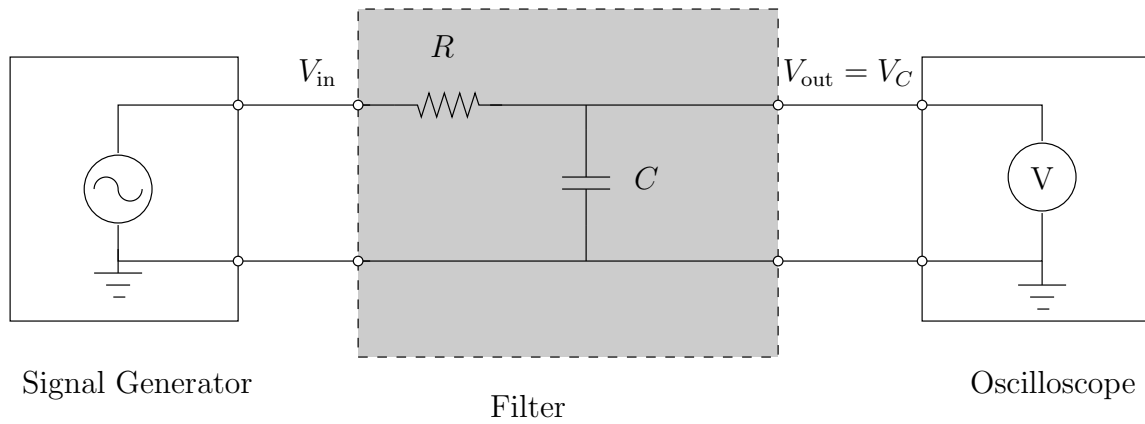


Figure 4.2: Simple RC circuit with explicitly drawn input and output.

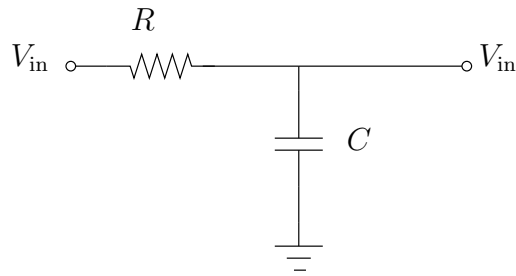


Figure 4.3: Simple RC circuits with implied input and output circuit elements. This circuit is known as an *integrator*.

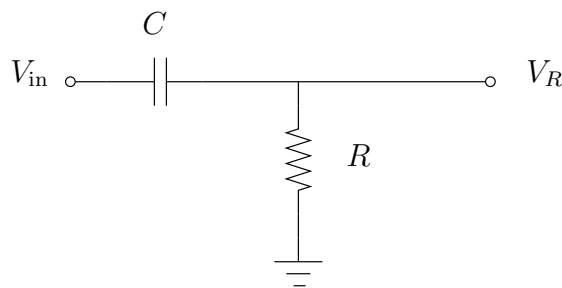


Figure 4.4: Simple RC circuits with implied input and output circuit elements. This circuit is known as a *differentiator*.

Reading

- Simpson, 2.3, 3.3–3.4 (Mixture of response to step-function changes and response to sinusoidal signals.)
- Horowitz & Hill, 1.12–1.15
- O’Malley, Chapter 8
- Nahvi & Edminster, Chapter 7 (Ignore material on inductors for now)
- Scherz, 2.20, 2.23.1–2.23.11, 2.34 (Ignore inductors for now)

Topic 5

Representation of Sinusoidal Functions with Complex Numbers and Phasors

5.1 Introduction

It is very useful to be able to represent sinusoidal functions in several ways, some of which exploit the relationship between sines and cosines and complex exponentials. The goal is to be able to convert easily between four representations of a sinusoidal function like those displayed in Fig. 5.1. If you see one representation, the others should be at your fingertips.

All of the representations in Fig. 5.1 refer to the same electrical oscillation: a cosine function with an amplitude of 0.3 V, a period $T = 20$ ms, a frequency $f = 1/T = 50$ Hz, and angular frequency $\omega = 2\pi f = 100\pi \simeq 314.16 \text{ s}^{-1}$, and a phase shift of $\phi = 45^\circ = \pi/4 \text{ rad} \simeq 0.79 \text{ rad}$. Representation A is a graphical representation like you might see on the face of an oscilloscope; Representation B is the standard algebraic representation of a cosine function with the appropriate amplitude, angular frequency, and phase. Representations C and D may not be as familiar to you — they display algebraic and graphical representations of complex numbers that are used to represent the oscillation.

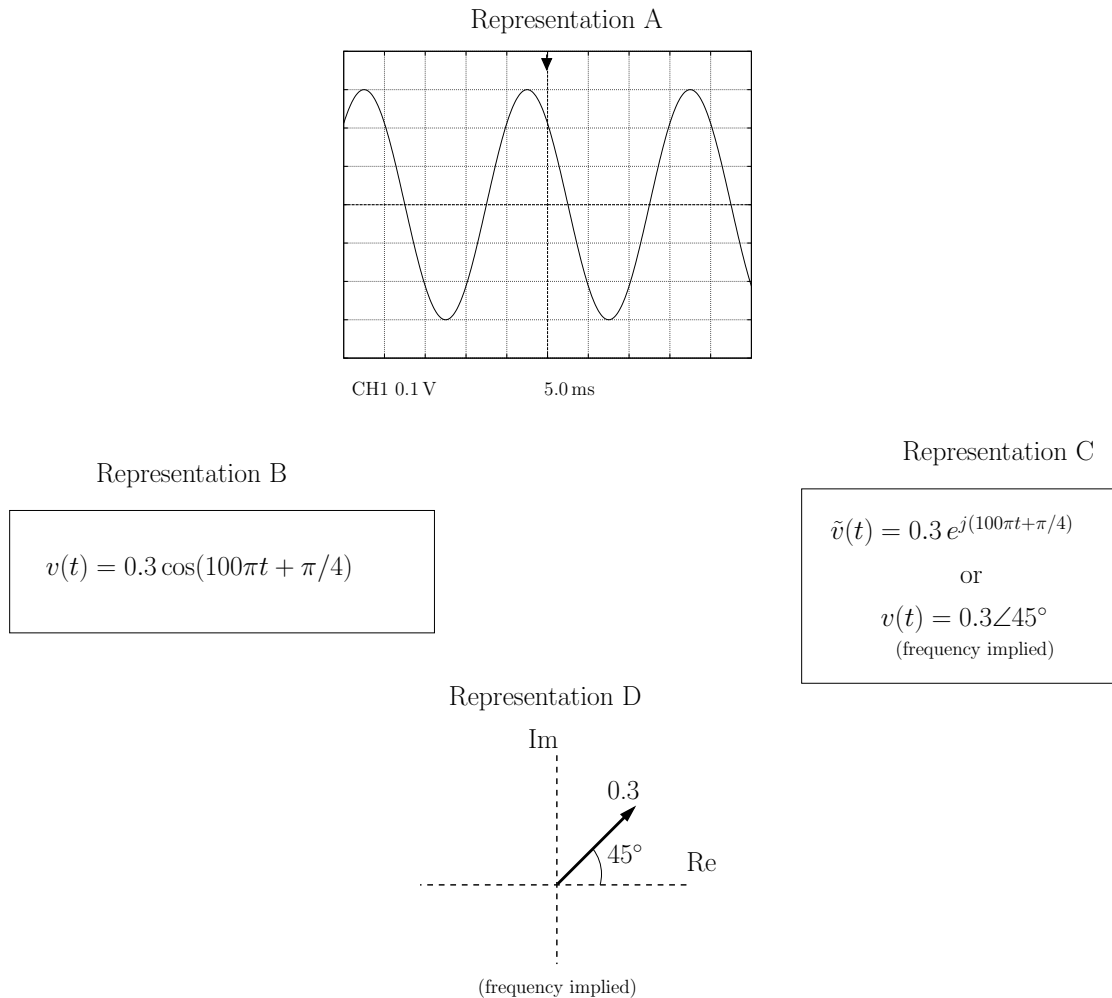


Figure 5.1: Four equivalent representations of the same sinusoidal function that might appear on an oscilloscope screen as in Representation A.

5.2 Review: Complex Numbers and the Complex Plane

The imaginary unit j is defined such that

$$j^2 = -1. \quad (5.1)$$

(In electronics and electrical engineering we use j instead of i to avoid confusion with current, which is often denoted by i .)

Complex numbers are numbers that are the linear combination of an imaginary

part and a real part:

$$z = a + jb, \quad (5.2)$$

where a and b are both real numbers. We say that the real part of the complex number z is a (or $\text{Re}(z) = a$) and the imaginary part is b (or $\text{Im}(z) = b$). Complex numbers can be plotted in the *complex plane*, where the real part of the number is plotted along the horizontal axis, and the imaginary part of the number is plotted along the vertical axis. For example, the complex number $z_1 = 4 + 3j$, with a real part of 4 and an imaginary part of 3, and the complex number $z_2 = 4 - 3j$, with a real part of 4 and an imaginary part of -3 , are plotted in Fig. 5.2.

The *complex conjugate* of a complex number is formed by taking all j 's and turning them into $-j$'s, e.g., if $z = a + jb$, then the complex conjugate, denoted with an asterisk, is $z^* = a - jb$. In the example illustrated in Fig. 5.2, z_2 is the complex conjugate of z_1 .

The *absolute value* of a complex number, $|z|$, often called the *modulus* is equal to the distance from the origin to the point as plotted in the complex plane. For a complex number $z = a + jb$, the absolute magnitude is

$$|z| = \sqrt{a^2 + b^2}.$$

It is useful to note the relationship

$$|z| = \sqrt{z \times z^*}.$$

5.3 Review: The complex exponential

Euler's relations,

$$e^{j\theta} = \cos \theta + j \sin \theta \quad \text{and} \quad e^{-j\theta} = \cos \theta - j \sin \theta \quad (5.3)$$

can be inverted to give

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \quad (5.4)$$

These relations suggest an alternative way to write complex numbers. For example, consider the quantity $Re^{j\theta}$. Using the relations above we find that

$$Re^{j\theta} = R(\cos \theta + j \sin \theta) = R \cos \theta + jR \sin \theta. \quad (5.5)$$

This is just a complex number with a real part of $R \cos \theta$ and an imaginary part of $R \sin \theta$. Writing complex numbers in the form of complex exponentials encourages

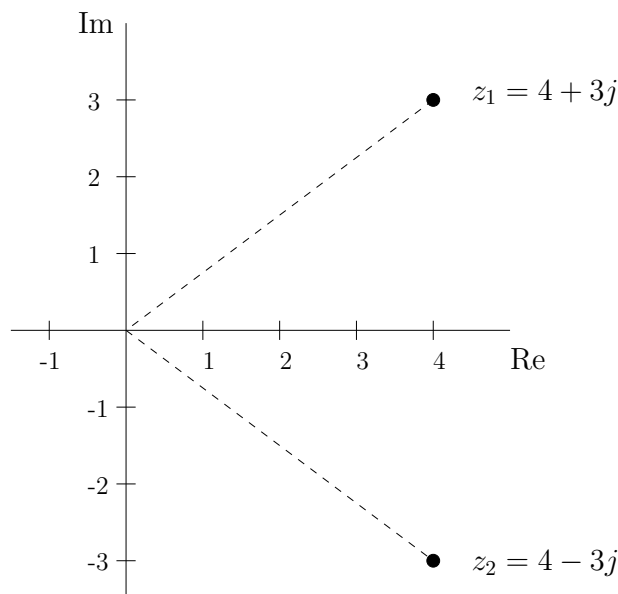


Figure 5.2: Plotting of complex numbers in the complex plane.

the geometric interpretation of complex numbers in the complex plane as is illustrated in Fig. 5.3 for the general complex number $z = R e^{j\theta} = a + jb$. Relationships between the quantities a , b , R , and θ are fundamentally geometric in nature, and they analogous to the relationships between the components of a vector and the vector's magnitude and direction.

5.4 Arithmetic of Complex Numbers

Consider two complex numbers,

$$z_1 = a_1 + jb_1 = R_1 e^{j\theta_1} \quad \text{and} \quad z_2 = a_2 + jb_2 = R_2 e^{j\theta_2}. \quad (5.6)$$

Addition: To add complex numbers simply add the real parts and imaginary parts separately. This is easiest in the “ $a + jb$ ” representation:

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2). \quad (5.7)$$

Multiplication: Complex numbers are easiest to multiply in the “ $R e^{j\theta}$ ” representation:

$$z_1 \times z_2 = R_1 R_2 e^{j(\theta_1 + \theta_2)}, \quad (5.8)$$

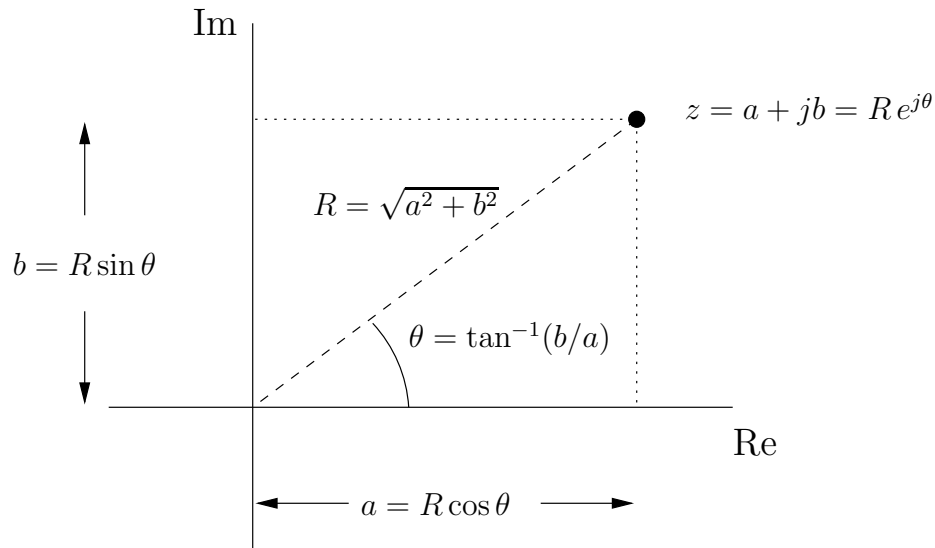


Figure 5.3: Geometric interpretation of complex exponential representation of complex numbers.

although it's not too bad as

$$\begin{aligned} z_1 \times z_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \end{aligned} \quad (5.9)$$

Here's an example of division:

$$\begin{aligned} \frac{3 + j4}{2 - 2j} &= \frac{5 e^{j \tan^{-1}(4/3)}}{\sqrt{8} e^{j \tan^{-1}(1)}} \\ &= \frac{5}{2\sqrt{2}} e^{j(\tan^{-1}(4/3) - \tan^{-1}(1))} \\ &\simeq 1.77 e^{0.142j} \end{aligned} \quad (5.10)$$

5.5 AC Signals and Complex Numbers

Consider a sinusoidal signal

$$v_1(t) = R_1 \cos \omega t. \quad (5.11)$$

This signal is the *real part* of the complex function

$$\tilde{v}_1 = R_1 e^{j\omega t} = R_1 \cos \omega t + jR_1 \sin \omega t. \quad (5.12)$$

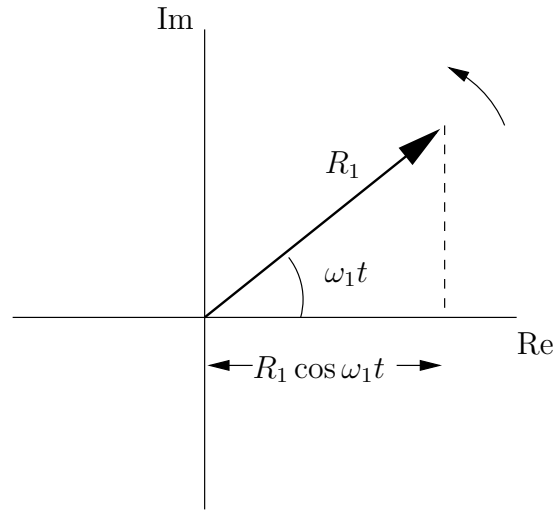


Figure 5.4: Rotating phasor whose real part corresponds to a real ac signal.

The complex quantity \tilde{v}_1 can be represented by a rotating *phasor* in the complex plane, and the real signal is the projection of the rotating phasor on the real (horizontal) axis. The phasor rotates counterclockwise as the angle $\omega_1 t$ increases with time as is illustrated in Fig. 5.4.

Complex representation of sinusoidal signals becomes very useful when you want to add signals that are out of phase with each other. (This is the kind of thing you need do with Kirchoff's loop rule in AC circuits.) For example, consider the addition of the signal $v_1(t)$ to a second signal

$$v_2(t) = R_2 \cos(\omega t + \phi_2). \quad (5.13)$$

This signal is the real part of

$$\tilde{v}_2 = R_2 e^{j(\omega t + \phi_2)} = (R_2 e^{j\phi_2}) e^{j\omega t} \quad (5.14)$$

At time $t = 0$ the phasors representing the two signals are illustrated in Fig. 5.5. The complex signal \tilde{v}_1 is entirely real at this time, and the signal \tilde{v}_2 is oriented at an angle ϕ_2 above the real axis. The sum of the two phasors representing \tilde{v}_1 and \tilde{v}_2 is also shown in the figure. As time increases from $t = 0$ all the phasors will rotate counterclockwise an angular frequency ω , but the lengths and relative orientations of the phasors will remain the same — the phasor representing \tilde{v}_2 will always be *ahead* of the phasor representing \tilde{v}_1 by an angle ϕ_2 , and the phasor representing the sum of the two signals will always be ahead of \tilde{v}_1 by the same angle α (and behind \tilde{v}_2 by an angle $\phi_2 - \alpha$). The angle α and the magnitude R_{tot} can be calculated using geometry and trigonometry in a manner that is analogous to the determination of the magnitude and direction when determining the sum of vectors.

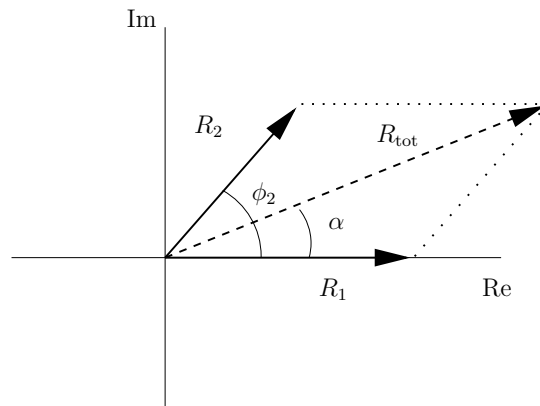


Figure 5.5: Phasors representing out of phase signals and their sum.

5.6 Problems

Reading

- For complex numbers and sinusoidal functions see French, *Vibrations and Waves*, pp. 10–15.
 - Simpson,
 - Horowitz & Hill,
 - O'Malley, Chapter 8
 - Nahvi & Edminster, Chapter 7
 - Scherz,

Topic 6

AC Circuits

6.1 Introduction

Previously we have dealt with the response of RC circuits to step-functions changes in potentials. Here we look at the response to continuous sinusoidal potentials.

For DC voltages and currents there is a direct proportionality between the voltage across a resistor and the current through the resistor, and the same proportionality (Ohm's Law) holds for ideal resistors with AC voltages and currents. Things are a little more complicated when we consider the voltage across a capacitor and the instantaneous current flowing onto the plates of a capacitor. For a capacitor

$$q = Cv$$

and if there is a sinusoidal voltage

$$v = A \cos(\omega t + \phi)$$

across the capacitor then the current flowing onto the capacitor must be

$$i = \frac{dq}{dt} = -\omega CA \sin(\omega t + \phi) = \omega CA \cos(\omega t + \phi + \pi/2).$$

Notice that the voltage v and the current i are not related by a simple multiplicative constant like they are in a resistor. The *magnitudes* of the voltage and the current are related by the multiplicative constant ωC , but there is a *phase difference* between v and i of $\pi/2$, or 90° .

For resistors we have

$$\frac{v}{i} = \text{real constant},$$

and that real constant is what we call the resistance. The complex number representation gives us a convenient way to deal with phase differences in AC circuits, and we'll show that for capacitors

$$\frac{\tilde{v}}{\tilde{i}} = \text{imaginary constant},$$

and we call the imaginary constant the *impedance*.

In complex form our AC voltage and current are

$$\tilde{v} = Ae^{j(\omega t + \phi)},$$

and

$$\tilde{i} = \omega CAe^{j(\omega t + \phi + \pi/2)}.$$

and the ratio is

$$\frac{\tilde{v}}{\tilde{i}} = \frac{1}{\omega Ce^{j\pi/2}} = \frac{1}{j\omega C} = \frac{-j}{\omega C}.$$

This complex ratio is the impedance of the capacitor, or

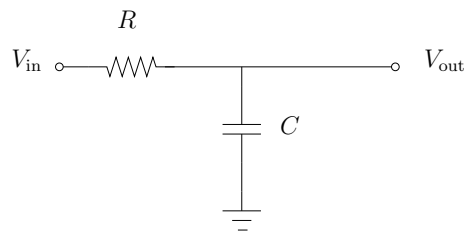
$$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}.$$

We now general Ohm's Law to

$$\tilde{v} = \tilde{i}Z,$$

and use it as before. Note that the impedance of a resistor is a pure real number, $Z_R = R$.

6.2 Low Pass Filter

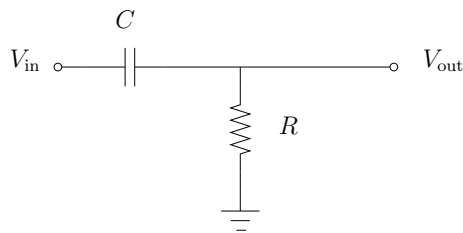


Treat this as a complex voltage divider:

$$\begin{aligned}\frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} &= \frac{Z_C}{Z_R + Z_C} \\ &= \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{j\phi_L},\end{aligned}\tag{6.1}$$

where $\phi_L = -\tan^{-1}(\omega RC)$.

6.3 High Pass Filter

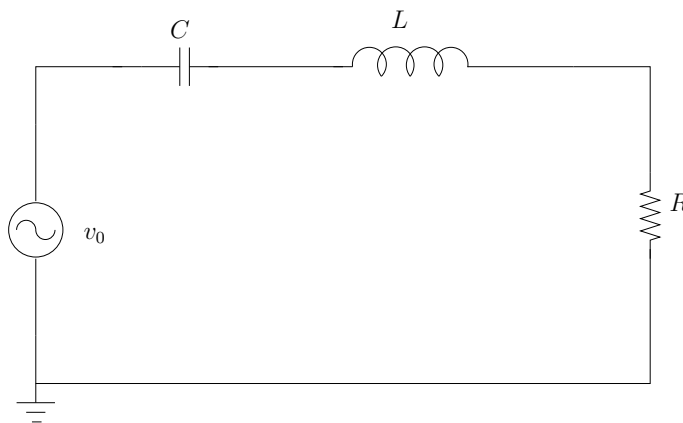


Treat this as a complex voltage divider:

$$\begin{aligned}\frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} &= \frac{Z_R}{Z_R + Z_C} \\ &= \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} e^{j\phi_H}\end{aligned}\tag{6.2}$$

where $\phi_H = \tan^{-1}(1/\omega RC)$.

6.4 RLC series circuit and resonance



Treat this as a complex voltage divider:

$$\begin{aligned}\frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} &= \frac{Z_R}{Z_R + Z_L + Z_C} \\ &= \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \\ &= \frac{R}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} e^{j\phi}\end{aligned}\tag{6.3}$$

where $\phi_H = -\tan^{-1}(\omega L/R - 1/\omega RC)$.

Reading

- Simpson, Chapter 2
- Horowitz & Hill, Chapter 1.18–1.22
- Scherz, 2.23.10–2.25, 2.30