Vehicular Traffic: A System of Interacting Particles Driven Far From Equilibrium*

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In recent years statistical physicists have developed *discrete* "particle-hopping" models of vehicular traffic, usually formulated in terms of *cellular automata*, which are similar to the microscopic models of interacting charged particles in the presence of an external electric field. Concepts and techniques of non-equilibrium statistical mechanics are being used to understand the nature of the steady states and fluctuations in these so-called "microscopic" models. In this brief review we explain, primarily to the nonexperts, these models and the physical implications of the results.

I. INTRODUCTION

Are you surprised to see an article on vehicular traffic in this special section of current science where physicists are supposed to report on some recent developments in the area of dynamics of nonequilibrium statistical systems? Aren't civil engineers (or, more specifically, traffic engineers) expected to work on traffic? Solving traffic problems would become easier if one knows the fundamental laws governing traffic flow and traffic jam. For almost half a century physicists have been trying to develope a theoretical framework of traffic science extending concepts and techniques of statistical physics [1–6]. The main aim of this brief review is to show how these attempts, particularly the recent ones, have led to deep insight in this frontier area of inter-disciplinary research.

The dynamical phases of systems driven far from equilibrium are counterparts of the stable phases of systems in equilibrium. Let us first pose some of the questions that statistical physicists have been addressing in order to discover the *fundamental laws* governing vehicular traffic. For example,
(i) What are the various *dynamical phases* of traffic? Does traffic exhibit phase-coexistence, phase transition, criticality or self-organized criticality and, if so, under which circumstances?
(ii) What is the nature of *fluctuations around the steady-states* of traffic?
(iii) If the initial state is far from a stationary state of the driven system, how does it *evolve with time* to reach a truely steady-state?
(iv) What are the effects of *quenched disorder* (i.e., time-independent disorder) on the answers to the questions posed in (i)-(iii) above?
The microscopic models of vehicular traffic can find *practical applications* in on-line traffic control systems as well as in the planning and design of transportation network.

There are two different conceptual frameworks for modellig vehicular traffic. In the "coarse-grained" fluid-dynamical description, the traffic is viewed as a compressible fluid formed by the vehicles but these individual vehicles do not appear explicitly in the theory. In contrast, in the "microscopic" models traffic is treated as *a system of interacting "particles" driven far from equilibrium* where attention is explicitly focussed on individual vehicles each of which is represented by a "particle"; the nature of the interactions among these particles is determined by the way the vehicles influence each others' movement. Unlike the particles in a gas, a driver is an intelligent agent who can "think", make individual decisions and "learn" from experience. Nevertheless, many general phenomena in traffic can be explained in general terms with these models provided the behavioural effects of the drivers are captured by only a few phenomenological parameters.

The conceptual basis of the older theoretical approaches are explained briefly in section II. Most of the "microscopic" models developed in the recent years are "particle-hopping" models which are usually formulated using the language of cellular automata (CA) [7]. The Nagel-Schreckenberg (NaSch) [8] model and the Biham-Middleton-Levine (BML) [9] model, which are the most popular CA models of traffic on idealized highways and cities, respectively, have been extended by several authors to develope more realistic models. Some of the most interesting aspects of these recent developments are discussed in the long sections III and IV. The similarities between various particle-hopping models of traffic and some other models of systems, which are also far from equilibrium, are pointed out in section V followed by the concluding section VI.

II. OLDER THEORIES OF VEHICULAR TRAFFIC

A. Fluid-dynamical Theories of vehicular traffic

In traffic engineering, the *fundamental diagram* depicts the relation between density c and the **flux** J, which is defined as the number of vehicles crossing a detector site per unit time [10]. Because of the conservation of vehicles, the local density $c(x;t)$ and local flux $J(x;t)$ satisfy the equation of continuity which is the analogue of the equation of continuity in the hydrodynamic theories of fluids. In the early works [12] it was assumed

(i) that the flux (or, equivalently, the velocity) is a function of the density and (ii) that, following any change in the local density, the local speed instantaneously relaxes to a magnitude consistent with the new density at the same location. However, for a more realistic description of traffic, in the recent fluid-dynamical treatments [13–15] of traffic an additional equation (the analogue of the Navier-Stokes equation for fluids), which describes the time-dependence of the velocity $V(x;t)$, has been considered. This approach, however, has its limitations; for example, viscosity of traffic is not a directly measureable quantity.

B. Kinetic theory of vehicular traffic

In the kinetic theory of traffic, one begins with the basic quantity $g(x,v,w;t)\ dx\ dv\ dw$ which is the number of vehicles, at time t, located between x and $x+dx$, having *actual* velocity between v and $v+dv$ and *desired* velocity between w and $w+dw$. In this approach, the fundamental dynamical equation is the analogue of the Boltzmann equation in the kinetic theory of gases [3]. Assuming reasonable forms of "relaxation" and "interaction", the problem of traffic is reduced to that of solving the Boltzmann-like equation, a formidable task, indeed [16–18]!

C. Car-following theories of vehicular traffic

In the car-following theories one writes, for each individual vehicle, an equation of motion which is the analogue of the Newton's equation for each individual particle in a system of interacting classical particles. In Newtonian mechanics, the acceleration may be regarded as the *response* of the particle to the *stimulus* it receives in the form of force which includes both the external force as well as those arising from its interaction with all the other particles in the system. Therefore, the basic philisophy of the car-following theories [1,2] can be summarized by the equation

$$[Response]_n \propto [Stimulus]_n \tag{1}$$

for the n-th vehicle ($n = 1,2,...$). The constant of proportionality in the equation (1) can be interpreted as a measure of the sensitivity coefficient of the driver; it indicates how strongly the driver responds to unit stimulus. Each driver can respond to the surrounding traffic conditions only by accelerating or decelerating the vehicle. The stimulus and the sensitivity factor are assumed to be functions of the position and speed of the vehicle under consideration and those of its leading vehicle. Different forms of the equations of motion of the vehicles in the different versions of the car-following models arise from the differences in their postulates regarding the nature of the stimulus. In general, the dynamical equations for the vehicles in the car-following theories are coupled non-linear differential equations [19–25] and thus, in this "microscopic" approach, the problem of traffic flow reduces to problems of nonlinear dynamics.

III. CELLULAR-AUTOMATA MODELS OF HIGHWAY-TRAFFIC

In the car-following models space is treated as a continuum and time is represented by a continuous variable t while velocities and accelerations of the vehicles are also real variables. However, most often, for numerical manipulations of the differential equations of the car-following models, one needs to discretize the continuous variables with appropriately chosen grids. In contrast, in the CA models of traffic not only time but also the position, speed, and acceleration of the vehicles are treated as *discrete* variables. In this approach, a lane is represented by a one-dimensional lattice. Each of the lattice sites represents a "cell" which can be either empty or occupied by at most one "vehicle" at a given instant of time (see fig.1).

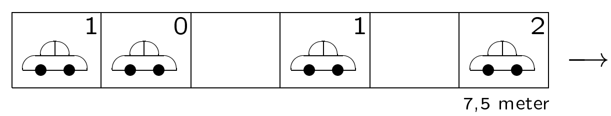

FIG. 1. A typical configuration in a CA model. The number in the upper right corner is the speed of the vehicle.

At each *discrete time* step $t \rightarrow t+1$, the state of the system is updated following a well defined prescription.

A. The Nagel-Schreckenberg model of highway traffic:

In the NaSch model, the speed V of each vehicle can take one of the $V_{max}+1$ allowed *integer* values $V = 0,1,...,V_{max}$. Suppose, X_n and V_n denote the position and speed, respectively, of the n-th vehicle. Then, $d_n = X_{n+1} - X_n$, is the gap in between the n-th vehicle and the vehicle in front of it at time t. At each time step $t \rightarrow t+1$, the arrangement of the N vehicles on a finite lattice of length L is updated *in parallel* according to the following "rules":

Step 1: Acceleration. If, $V_n < V_{max}$, the speed of the n-th vehicle is increased by one, but V_n remains unaltered if $V_n = V_{max}$, i.e., $V_n \rightarrow min(V_n+1, V_{max})$.

Step 2: Deceleration (due to other vehicles). If $d_n \leq V_n$, the speed of the n-th vehicle is reduced to $d_n - 1$, i.e., $V_n \rightarrow min(V_n, d_n - 1)$.

Step 3: Randomization. If $V_n > 0$, the speed of the n-th vehicle is decreased randomly by unity with probability

p but V_n does not change if $V_n = 0$, i.e., $V_n \to max(V_n - 1, 0)$ with probability p.
Step 4: Vehicle movement. Each vehicle is moved forward so that $X_n \to X_n + V_n$.

The NaSch model is a minimal model in the sense that all the four steps are necessary to reproduce the basic features of real traffic; however, additional rules need to be formulated to capture more complex situations. The step 1 reflects the general tendency of the drivers to drive as fast as possible, if allowed to do so, without crossing the maximum speed limit. The step 2 is intended to avoid collision between the vehicles. The randomization in step 3 takes into account the different behavioural patterns of the individual drivers, especially, nondeterministic acceleration as well as overreaction while slowing down; this is crucially important for the spontaneous formation of traffic jams. So long as $p \neq 0$, the NaSch model may be regarded as stochastic CA [7]. For a realistic description of highway traffic [8], the typical length of each cell should be about 7.5m and each time step should correspond to approximately 1 sec of real time when $V_{max} = 5$.

The update scheme of the NaSch model is illustrated with a simple example in fig.2.

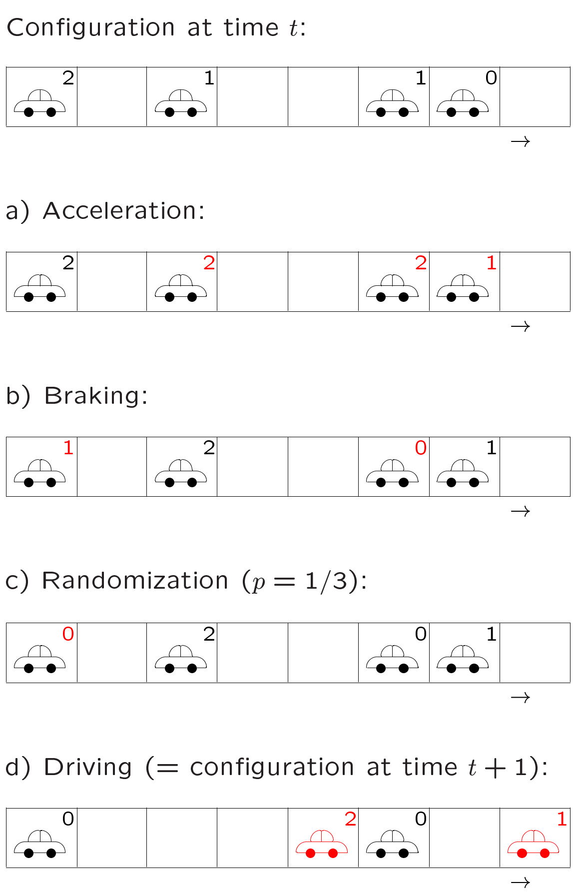

FIG. 2. Step-by-step example for the application of the update rules. We have assumed $V_{max} = 2$ and $p = 1/3$. Therefore on average one third of the cars qualifying will slow down in the randomization step.

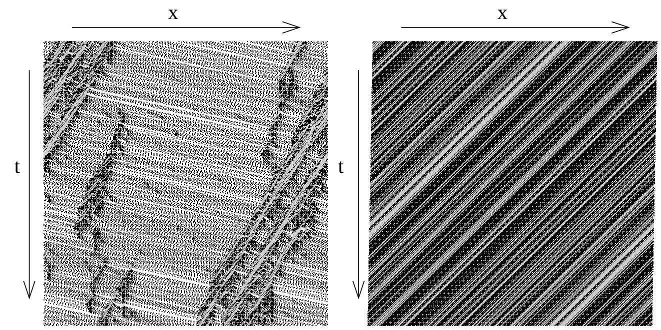

FIG. 3. Typical space-time diagrams of the NaSch model with $V_{max} = 5$ and (a) $p = 0.25, c = 0.20$, (b) $p = 0.0, c = 0.5$. Each horizontal row of dots represents the instantaneous positions of the vehicles moving towards right while the successive rows of dots represent the positions of the same vehicles at the successive time steps.

Space-time diagrams showing the time evolutions of the NaSch model demonstrate that no jam is present at sufficiently low densities, but spontaneous fluctuations give rise to traffic jams at higher densities (fig.3(a)). From the fig.3(b) it should be obvious that the *intrinsic stochasticity* of the dynamics [8], arising from non-zero p, is essential for triggering the jams [8,26].

The use of *parallel* dynamics is also important. In contrast to a random sequential update, it can lead to a chain of overreactions. Suppose, a vehicle slows down due the randomization step. If the density of vehicles is large enough this might force the following vehicle also to brake in the deceleration step. In addition, if p is not too small, it might brake even further in Step 3. Eventually this can lead to the stopping of a vehicle, thus creating a jam. This mechanism of spontaneous jam formation is rather realistic and cannot be modelled by the random sequential update.

B. Relation between the NaSch model and ASEP

In the NaSch model with $V_{max} = 1$ every vehicle moves forward with probability $q = 1 - p$ in the time step $t + 1$ if the site immediately in front of it were empty at the time step t; this, is similar to the fully asymmetric simple exclusion process (**ASEP**) [27–29] where a randomly chosen particle can move forward with probablity q if the site immediately in front is empty. But, updating is done in parallel in the NaSch model whereas that in the ASEP is done in a random sequential manner. Nevertheless, the special case of $V_{max} = 1$ for the NaSch model achieves special importance from the fact that so far it has been possible to derive exact analytical results for the NaSch model only in the special limits (a) $V_{max} = 1$ and arbitrary p and (b) $p = 0$ and arbitrary V_{max}.

C. NaSch model in the deterministic limits

If $p = 0$, the system can self-organize so that at low densities every vehicle can move with V_{max} and the corresponding flux is cV_{max}; this is, however, possible only if enough empty cells are available in front of every vehicle, i.e., for $c \leq c_m^{det} = (V_{max} + 1)^{-1}$ and the corresponding maximum flux is $J_{max}^{det} = V_{max}/(V_{max} + 1)$. On the other hand, for $c > c_m^{det}$, the flow is limited by the density of holes. Hence, the fundamental diagram in the deterministic limit $p = 0$ of the NaSch model (for any arbitrary V_{max}) is given by the *exact* expression $J = min(cV_{max}, (1 - c))$.

Aren't the properties of the NaSch model with maximum allowed speed V_{max}, in the deterministic limit $p = 1$, exactly identical to those of the same model with maximum allowed speed $V_{max} - 1$? The answer to the question posed above is: NO; if $p = 1$, all random initial states lead to $J = 0$ in the stationay state of the NaSch model irrespective of V_{max} and c!

D. Analytical Theory for the NaSch Model

In the "site-oriented" theories one describes the state of the finite system of length L by completely specifying the state of each *site*. In contrast, in the "car-oriented" theories the state of the traffic system is described by specifying the positions and speeds of all the N vehicles in the system. In the naive mean-field approximation one treats the probabilities of occupation of the lattice sites as independent of each other. In this approximation, for example, the steady-state flux for the NaSch model with $V_{max} = 1$ and periodic boundary conditions, one gets [30]

$$J = qc(1 - c) \tag{2}$$

It turns out [30] that the naive mean-field theory underestimates the flux for all V_{max}. Curiously, if instead of parallel updating one uses the random sequential updating, the NaSch model with $V_{max} = 1$ reduces to the ASEP for which the equation (2) is known to be the *exact* expression for the corresponding flux (see, e.g., [8])!

What are the reasons for these differences arising from parallel updating and random sequential updating? There are "garden of Eden" (GoE) states (dynamically forbidden states) [31] of the NaSch model which cannot be reached by the parallel updating whereas no state is dynamically forbidden if the updating is done in a random sequential manner.

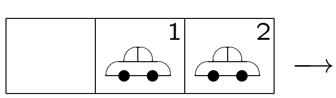

FIG. 4. A GoE state for the NaSch model with $V_{max} \geq 2$.

For example, the configuration shown in fig.4 is a GoE state* because it could occur at time t only if the two vehicles occupied the same cell simultaneously at time $t-1$. The naive mean-field theory mentioned above does not exclude the GoE states. The exact expression, given in the next subsection, for the flux in the steady-state of the NaSch model with $V_{max} = 1$ can be derived by merely excluding these states from consideration in the naive mean-field theory [31], thereby indicating that the only source of correlation in this case is the parallel updating. But, for $V_{max} > 1$, there are other sources of correlation because of which exclusion of the GoE states merely improves the naive mean-field estimate of the flux but does not yield exact results [31].

A systematic improvement of the naive mean-field theory of the NaSch model has been achieved by incorporating short-ranged correlations through cluster approximations. We define a n-cluster to be a collection of n successive sites. In the general n-cluster approximation, one divides the lattice into "clusters" of length n such that two neighbouring clusters have $n-1$ sites in common (see fig.5).

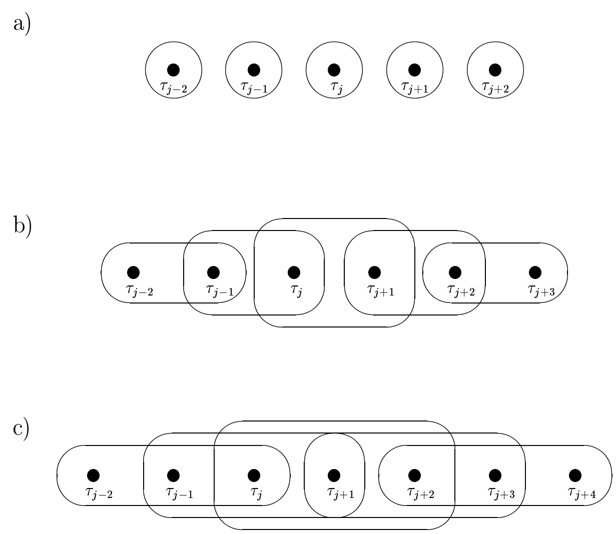

FIG. 5. Decomposition of a lattice into (a) 1-clusters (b) 2-clusters and (c) 3-clusters in the cluster-theoretic approach to the NaSch model.

If $n = 1$, then the 1-cluster approximation can be regarded as the naive mean-field approximation. You can easily verify, for example, in the special case of $V_{max} = 1$, that the state of the 2-cluster at time $t+1$ depends on the state of the 4-cluster at time t, which, in turn, depends on the state of a larger cluster at time $t - 1$ and, so on.

* The configuration shown in fig.1 is also a GoE state!

Therefore, one needs to make an approximation to truncate this hierarchy in a sensible manner. For example, in the 2-cluster approximation for the NaSch model with $V_{max} = 1$, the 4-cluster probabilities are approximated in terms of an appropriate product of 2-cluster probabilities. Thus, in the n-cluster approximation [30] a cluster of n neighbouring cells are treated exactly and the cluster is coupled to the rest of the system in a self-consistent way.

Carrying out the 2-cluster calculation [30] for $V_{max} = 1$ one not only finds an effective particle-hole attraction (particle-particle repulsion), but also obtains the exact result

$$J(c,p) = \frac{1}{2}[1 - \sqrt{1 - 4qc(1-c)}] \tag{3}$$

for the corresponding flux. But one gets only approximate results from the 2-cluster calculations for all $V_{max} > 1$ (see [32] for higher order cluster calculations for $V_{max} = 2$ and comparison with computer simulation data).

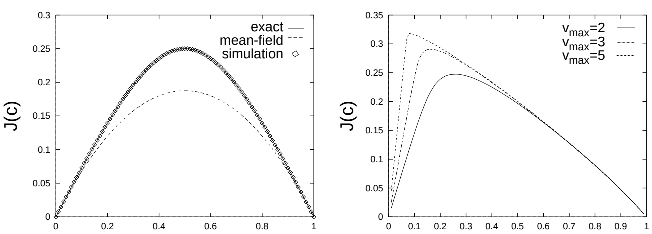

FIG. 6. The fundamental diagram in the NaSch model for (a) $V_{max} = 1$ and (b) $V_{max} > 1$, both for $p = 0.25$. The data for all $V_{max} > 1$ have been obtained through computer simulations.

Let us explain the physical origin of the generic shape of the fundamental diagrams shown in fig.6. At sufficiently low density of vehicles, practically "free flow" takes place whereas at higher densities traffic becomes "congested" and traffic jams occur. So long as c is sufficiently small, the average speed $\langle V \rangle$ is practically independent of c as the vehicles are too far apart to interact mutually. However, a faster monotonic decrease of $\langle V \rangle$ with increasing c takes place when the forward movement of the vehicles is strongly hindered by others because of the reduction in the average separation between them. Because of this trend of variation of $\langle V \rangle$ with c, the flux $J = \langle cV \rangle$ exhibits a maximum [10] at c_m; for $c < c_m$, increasing c leads to increasing J whereas for $c > c_m$ sharp decrease of $\langle V \rangle$ with increase of c leads to the overall decrease of J.

An interesting feature of the expression (3) is that the flux is invariant under charge conjugation, i.e., under the operation $c \to (1 - c)$ which interchanges particles and holes. Therefore, the fundamental diagram is symmetric about $c = 1/2$ when $V_{max} = 1$ (see fig.6a). Although this symmetry breaks down for all $V_{max} > 1$ (see fig.6b), the corresponding fundamental diagrams appear more realistic. Moreover, for given p, the magnitude of c_m decreases with increasing V_{max} as the higher is the V_{max} the longer is the effective range of interaction of the vehicles (see fig.6b). Furthermore, for $V_{max} = 1$, flux merely decreases with increasing p (see eqn(3)), but remains symmetric about $c = 1/2 = c_m$. On the other hand, for all $V_{max} > 1$, increasing p not only leads to smaller flux but also lowers c_m.

E. Spatio-temporal organization of vehicles

The distance from a selected point on the lead vehicle to the same point on the following vehicle is defined as the *distance-headway* (**DH**) [10]. In order to get information on the spatial organization of the vehicles, one can calculate the DH distribution $\mathcal{P}_{dh}(\Delta X)$ by following either a site-oriented approach [34] or a car-oriented approach [33] if $\Delta X_j = X_j - X_{j-1}$, i.e., if the number of empty lattice sites in front of the j-th vehicle is identified as the corresponding DH. At moderately high densities, $\mathcal{P}_{dh}(\Delta X)$ exhibits two peaks; the peak at $\Delta X = 1$ is caused by the jammed vehicles while that at a larger ΔX corresponds to the most probable DH in the free-flowing regions.

The *time-headway* is defined as the time interval between the departures (or arrivals) of two successive vehicles recorded by a detector placed at a fixed position on the highway [10]. The time-headway distribution contains information on the temporal organization. Suppose, $\mathcal{P}_m(t_1)$ is the probability that the following vehicle takes time t_1 to reach the detector, moving from its initial position where it was located when the leading vehicle just left the detector site. Suppose, after reaching the detector site, the following vehicle waits there for $\tau - t_1$ time steps, either because of the presence of another vehicle in front of it or because of its own random braking; the probability for this event is denoted by $Q(\tau - t_1|t_1)$. The distribution $\mathcal{P}_{th}(\tau)$, of the time-headway τ, can be obtained from [35,36] $\mathcal{P}_{th}(\tau) = \sum_{t_1=1}^{\tau-1} \mathcal{P}_m(t_1) Q(\tau - t_1|t_1)$. The most-probable time-headway, when plotted against the density, exhibits a minimum [36]; this is consistent with the well known exact relation $J = 1/T_{av}$ between flux and the average time-headway, T_{av}.

Is there a phase transition from "free-flowing" to "congested" dynamical phase of the NaSch model? No satisfacory order parameter has been found so far [37,38], except in the deterministic limit [39]. The possibility of the existence of any critical density in the NaSch model is ruled out by the observations [37,38,32,40] that, for all nonzero p, (a) the equal-time correlation function decays exponentially with separation, and (b) the relaxation time and lifetimes of the jams remain finite. This minimal model of highway traffic also does not exhibit any first order phase transition and two-phase co-existence [35].

F. Extensions of the NaSch model and practical applications

In recent years some other minimal models of traffic on highways have been developed by modifying the updating rules of the NaSch model [41–43]. In the cruise control limit of the NaSch model [44] the randomization step is applied only to vehicles which have a velocity $V < V_{max}$ after step 2 of the update rule. Vehicles moving with their desired velocity V_{max} are not subject to fluctuations. This is exactly the effect of a cruise-control which automatically keeps the velocity constant at a desired value. Interestingly, the cruise-control limit of the NaSch model exhibits self-organized criticality [45,46]. Besides, a continuum limit of the NaSch model has also been considered [47].

The vehicles which come to a stop because of hindrance from the leading vehicle may not be able to start as soon as the leading vehicle moves out of its way; it may start with a probability $q_s < 1$. When such possibilities are incorporated in the NaSch model, the "slow-to-start" rules [48–52] can give rise to metastable states of very high flux and hysteresis effects as well as phase separation of the traffic into a "free-flowing" phase and a "mega-jam".

The bottleneck created by quenched disorder of the *highway* usually slows down traffic and can give rise to jams [53,35] and phase segregation [54,55]. However, a different type of quenched disorder, introduced by assigning randomly different braking probabilities p to different drivers in the NaSch model, can have more dramatic effects [56,57] which are reminiscent of "Bose-Einstein-like condensation" in the fully ASEP where particle-hopping rates are quenched random variables [58,59]. In such "Bose-Einstein-like condensed" state finite fraction of the empty sites are "condensed" in front of the slowest vehicle (i.e., the driver with highest p).

Several attempts have been made to generalize the NaSch model to describe traffic on multi-lane highways and to simulate traffic on real networks in and around several cities [60]. For *planning and design* of the transportation network [66], for example, in a metropolitan area [61–63], one needs much more than just *micro-simulation* of how vehicles move on a linear or square lattice under a specified set of vehicle-vehicle and road-vehicle interactions. For such a simulation, to begin with, one needs to specify the roads (including the number of lanes, ramps, bottlenecks, etc.) and their intersections. Then, times and places of the activities, e.g., working, shopping, etc., of individual drivers are planned. Micro-simulations are carried out for all possible different routes to execute these plans; the results give informations on the efficiency of the different routes and these informations are utilized in the designing of the transportation network [64–66]. Some socio-economic questions as well as questions on the environmental impacts of the planned transportation infrastructure also need to be addressed during such planning and design.

IV. CELLULAR-AUTOMATA MODELS OF CITY-TRAFFIC

A. The Biham-Middleton-Levin model of city traffic and its generalizations

In the BML model [9], each of the sites of a square lattice represent the crossing of a east-west street and a north-south street. All the streets parallel to the $\hat{X}$-direction of a Cartesian coordinate system are assumed to allow only *single-lane* east-bound traffic while all those parallel to the $\hat{Y}$-direction allow only single-lane north-bound traffic. In the initial state of the system, vehicles are randomly distributed among the streets. The states of east-bound vehicles are updated in parallel at every odd discrete time step whereas those of the north-bound vehicles are updated in parallel at every even discrete time step following a rule which is a simple extension of the fully ASEP: a vehicle moves forward by one lattice spacing if and only if the site in front is empty, otherwise the vehicle does not move at that time step.

Computer simulations demonstrate that a *first order* phase transition takes place in the BML model at a finite non-vanishing density c_*, where the average velocity of the vehicles vanishes *discontinuously* signalling complete jamming; this jamming arises from the mutual blocking of the flows of east-bound and north-bound traffic at various different crossings [67,68]. Note that the dynamics of the BML model is fully *deterministic* and the randomness arises only from the *random initial conditions* [69].

As usual, in the naive mean-field approximation one neglects the correlations between the occupations of different sites [70]. However, if you are not interested in detailed information on the "structure" of the dynamical phases, you can get a mean-field estimate of c_* by carrying out a back-of-the-envelope calculation [71–73]. In the symmetric case $c_x = c_y$, for which $v_x = v_y = v$, $c = c_* \simeq 0.343$.

The BML model has been extended to take into account the effects of (i) asymmetric distribution of the vehicles [71], i.e., $c_x \neq c_y$, (ii) overpasses or two-level crossings [72] that are represented by specifically identified sites each of which can accomodate upto a maximum of two vehicles simultaneously, (iii) faulty traffic lights [74] (iv) static hindrances or road blocks or vehicles crashed in traffic accident, i.e., stagnant points [75,76], (v) stagnant street where the local density c_s of the vehicles is initially higher than that in the other streets [77] (vi) jam-avoiding drive [78] of vehicles to a neighbouring street, parallel to the original direction, to avoid getting blocked by other vehicles in front, (vii) turning of the vehicles from east-bound (north-bound) to north-bound (east-bound) streets [79]. (viii) a single north-bound street cutting across east-bound streets [80] (ix) more realistic description of junctions of perpendicular streets [81,82], (x) green-waves [83].

B. Marriage of NaSch and BML models

At first sight the BML model may appear very unrealistic because the vehicles seem to hop from one crossing to the next. However, it may not appear so unrealistic if each unit of discrete time interval in the BML model is interpreted as the time for which the traffic lights remain green (or red) before switching red (or green) simultaneously in a synchronized manner, and over that time scale each vehicle, which faces a green signal, gets an opportunity to move from j-th crossing to the $j+1$-th (or, more generally [84], to the $j+r$-th where $r > 1$).

However, if one wants to develope a more detailed "fine-grained" description then one must first decorate each bond [85] with $D-1$ ($D > 1$) sites to represent $D-1$ cells in between each pair of successive crossings thereby modelling each segment of the streets in between successive crossings in the same manner in which the entire highway is modelled in the NaSch model. Then, one can follow the prescriptions of the NaSch model for describing the positions, speeds and accelerations of the vehicles [82,86] as well as for taking into account the interactions among the vehicles moving along the same street. Moreover, one should flip the color of the signal periodically at regular interval of T ($T >> 1$) time steps where, during each unit of the discrete time interval every vehicle facing green signal should get an opportunity to move forward from one cell to the next. Such a CA model of traffic in cities has, indeed, been proposed very recently [87] where the rules of updating have been formulated in such a way that, (a) a vehicle approaching a crossing can keep moving, even when the signal is red, until it reaches a site immediately in front of which there is either a halting vehicle or a crossing; and (b) no grid-locking would occur in the absence of random braking.

A phase transition from the "free-flowing" dynamical phase to the completely "jammed" phase has been observed in this model at a vehicle density which depends on D and T. The intrinsic stochasticity of the dynamics, which triggers the onset of jamming, is similar to that in the NaSch model, while the phenomenon of complete jamming through self-organization as well as the final jammed configurations (fig.7) are similar to those in the BML model. This model also provides a reasonable time-dependence of the average speeds of the vehicles in the "free-flowing" phase [87].

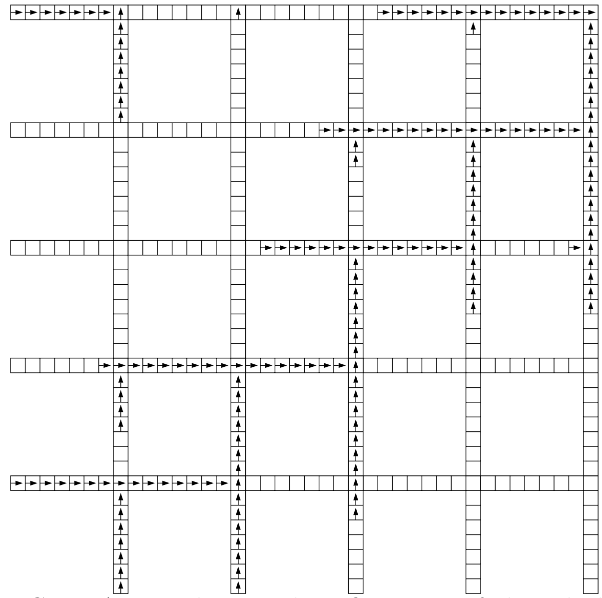

FIG. 7. A typical jammed configuration of the vehicles. The east-bound and north-bound vehicles are represented by the symbols $\rightarrow$ and $\uparrow$, respectively. ($N = 5, D = 8$)

V. RELATION WITH OTHER SYSTEMS AND PHENOMENA

You must have noticed in the earlier sections that some of the models of traffic are non-trivial generalizations or extensions of the ASEP, the simplest of the *driven-dissipative* systems which are of current interest in non-equilibrium statistical mechanics [28]. Some similarities between these systems and a dynamical model of protein synthesis has been pointed out [88]. Another driven-dissipative system, which is also receiving wide attention of physicists in recent years, is the granular material flowing through a pipe [4,5]. There are some superficial similarities between the clustering of vehicles on a highway and particle-particle (and particle-cluster) aggregation process [89].

The NaSch model with $V_{max} = 1$ can be mapped onto stochastic growth models of one-dimensional surfaces in a two-dimensional medium. Particle (hole) movement to the right (left) correspond to local forward growth of the surface via particle deposition. In this scenario a particle evaporation would correspond to a particle (hole) movement to the left (right) which is not allowed in the NaSch model. It is worth pointing out that any quenched disorder in the rate of hopping between two adjacent sites would correspond to *columnar* quenched disorder in the growth rate for the surface [55].

Inspired by the recent success in theoretical studies of traffic, some studies of information traffic on the computer network (internet) have also been carried out [90–92].

VI. SUMMARY AND CONCLUSION:

Nowadays the tools of statistical mechanics are increasingly being used to study self-organization and emergent collective behaviour of *complex systems* many of which, including vehicular traffic, fall outside the traditional domain of physical systems. However, as we have shown in this article, a strong theoretical foundation of traffic science can be built on the basic principles of statistical mechanics. In this brief review we have focussed attention mainly on the progress made in the recent years using "particle-hopping" models, formulated in terms of cellular automata, and compared these with several other similar systems.

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