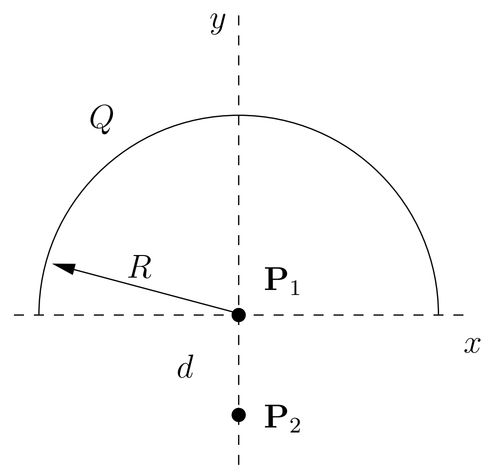
PHYS 333 — Final Exam
Monday December 15, 2014

Name:________________________

1. (20 pts) Consider the illustrated circular arc of radius R with charge Q uniformly distributed along the arc. (The arc lies in the x-y plane.)



   (a) Calculate the electric field at the origin ($\mathbf{P}_1$).

   (b) Determine an expression for the electric field at point $\mathbf{P}_2$, a distance d below the origin on the y-axis. You may leave your answer in the form of an integral which could be evaluated by a computer or calculator.

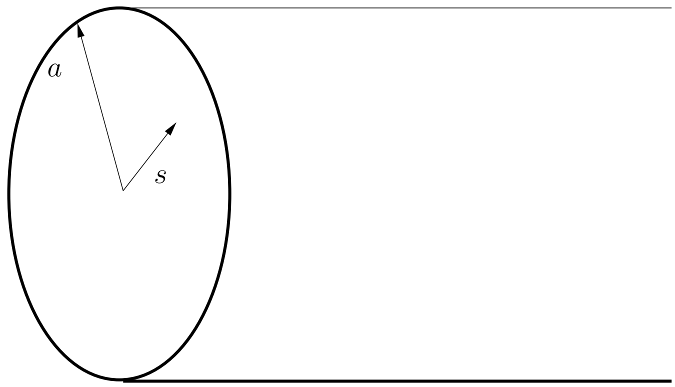
2. (10 pts) When you polarize a neutral dielectric, the charges moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b (surface bound charge density) and ρ_b (volume bound charge density). Prove that the total bound charge vanishes.

3. (20 pts) *Begin* a general solution of Laplace's equation, $\nabla^2 V = 0$, in cylindrical coordinates (s, ϕ, and z) using the separation-of-variables technique. **Assume that there is enough symmetry that V is independent of z.**

   (a) Determine the ordinary differential equations that must be solved for your product functions.

   (b) Solve *one* of the differential equations (your choice). State any restrictions on the undetermined constants in the general solution of the differential equation.

4. (20 pts) An infinite cylinder of radius a carries a volume charge density $\rho = ks^2$, where k is a constant and s is the perpendicular distance from the axis of the cylinder. Determine the electric field everywhere.



5. (10 pts) The electric field due to a charge density ρ can be written as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

(a) In this expression, the quantities $\mathbf{E}$, ρ, and $\boldsymbol{\mathscr{r}}$, depend on the vectors $\mathbf{r}$ and $\mathbf{r}'$ defined by Griffiths (and used throughout this course). In the following, circle the correct functional dependence for $\mathbf{E}$, ρ, and $\boldsymbol{\mathscr{r}}$.

$\mathbf{E}$:	$\mathbf{E}(\mathbf{r})$	$\mathbf{E}(\mathbf{r}')$	$\mathbf{E}(\mathbf{r}, \mathbf{r}')$
ρ :	$\rho(\mathbf{r})$	$\rho(\mathbf{r}')$	$\rho(\mathbf{r}, \mathbf{r}')$
$\boldsymbol{\mathscr{r}}$:	$\boldsymbol{\mathscr{r}}(\mathbf{r})$	$\boldsymbol{\mathscr{r}}(\mathbf{r}')$	$\boldsymbol{\mathscr{r}}(\mathbf{r}, \mathbf{r}')$

(b) We can use the integral at the top of the page to calculate $\mathbf{E}$, and in electrostatics we know that

$$\vec{\nabla} \times \mathbf{E} = 0.$$

In this expression the operator ∇ can be represented as (circle one):

$$\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \qquad \vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'} \right)$$

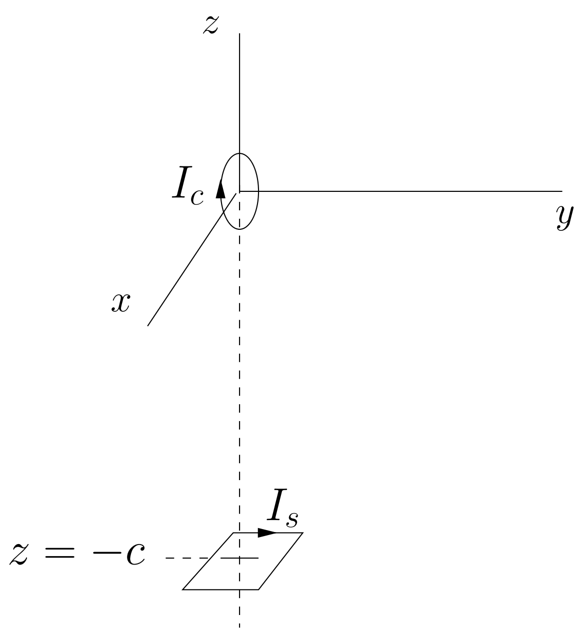
Justify your answer in a sentence or two.

6. (10 pts) An electric field is given by

$$\mathbf{E}(\mathbf{r}) = -(2x + 4y)\,\hat{\mathbf{x}} - (4x + 2z^3)\,\hat{\mathbf{y}} - 6yz^2\,\hat{\mathbf{z}}.$$

Calculate the potential difference between the points $(0, 0, 0)$ and $(1, 1, 0)$.

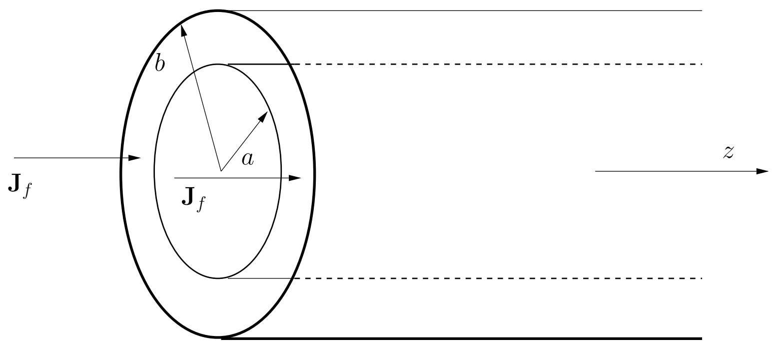
7. (20 pts) Consider the illustrated current loops. Assume that the radius of the circle is a and the sides of the square are length b; both a and b are assumed to be much smaller than c. The circle is situated in the x-z plane, and the square is in a plane parallel to the x-y plane.



(a) Calculate the torque on the square loop.

(b) If the square loop is free to rotate, describe its equilibrium orientation.

8. (20 pts) A uniform **free** current volume density $\mathbf{J}_\mathrm{f} = J_0\,\hat{\mathbf{z}}$ flows down an infinite cylindrical sheath of linear magnetic material with magnetic susceptibility χ_m. The inner radius of the sheath is a, and the outer radius is b.



(a) Determine the magnetic field $\mathbf{B}$ everywhere.

(b) Determine all bound currents.

(c) What is the net bound current flowing in the z-direction?

9. The electric and magnetic fields in a certain electromagnetic plane wave in free space are given by

$$\mathbf{E}(\mathbf{r},t) = E_0 \cos(ky + \omega t)\,\hat{\mathbf{z}} \tag{1}$$

$$\mathbf{B}(\mathbf{r},t) = -\frac{1}{c} E_0 \cos(ky + \omega t)\,\hat{\mathbf{x}} \tag{2}$$

(a) In what direction is the wave traveling? Circle one.

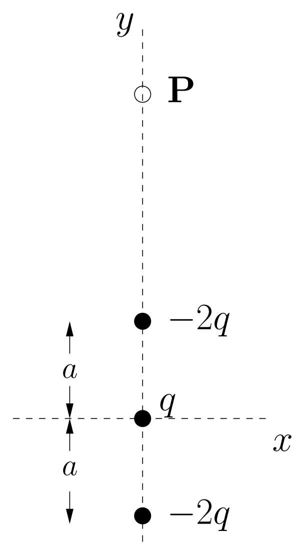
$$\hat{\mathbf{x}} \quad -\hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad -\hat{\mathbf{y}} \quad \hat{\mathbf{z}} \quad -\hat{\mathbf{z}}$$

(b) In what direction is the wave polarized? Circle one.

$$\hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad \hat{\mathbf{z}}$$

(c) Verify that this wave satisfies all four of Maxwell's equations.

10. (20 pts) Consider the illustrated arrangement of three charges.



(a) Write an exact expression for the potential due to these charges at point **P** on the y-axis.

(b) Find an approximation to your potential that is valid for points on the y-axis that are far away from the charges. (Use the binomial approximation and include the first **two** non-vanishing terms in your approximation.)

(c) Interpret your result in a sentence or two.

Equations

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \longleftrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da' \longleftrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$$\mathbf{I} = \lambda \mathbf{v}$$

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \longrightarrow \sigma \mathbf{v}$$

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \longrightarrow \rho \mathbf{v} \longleftrightarrow I = \int \mathbf{J} \cdot d\mathbf{a}$$

$$\mathbf{F} = q\,[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$

$$\mathbf{F}_{\text{mag}} = \int I(d\mathbf{l} \times \mathbf{B}) \longleftrightarrow \int (\mathbf{K} \times \mathbf{B})\, da \longleftrightarrow \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

$$\mathbf{m} = I \int d\mathbf{a}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}\right) = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\,]$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$$

$$\mathbf{J} = \sigma \mathbf{E}$$

$$\mathcal{E} \equiv \oint \mathbf{f}_s \cdot d\mathbf{l} \longrightarrow \oint (\mathbf{f}_{\text{elec}} + \mathbf{f}_{\text{mag}}) \cdot d\mathbf{l}$$

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{a} \longleftrightarrow \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\left(I_{\text{enc}} + \epsilon_0\frac{\partial}{\partial t}\int \mathbf{E}\cdot d\mathbf{a}\right) \longleftrightarrow \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{J}_d \equiv \epsilon_0\frac{\partial \mathrm{E}}{\partial t}$$

$$\Phi = LI \qquad \text{and} \qquad \Phi = MI$$

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$W = \frac{1}{2}LI^2$$

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad \nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Traveling Plane Wave Solutions:

$$\mathbf{E}(\mathbf{r},t) = E_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\,\hat{\mathbf{n}} \qquad \mathbf{B}(\mathbf{r},t) = \frac{E_0}{c}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\,(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}} \qquad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{c}\,\tilde{E}_0\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,(\hat{\mathbf{k}}\times\hat{\mathbf{n}}) = \frac{1}{c}\,\hat{\mathbf{k}}\times\tilde{\mathbf{E}}(\mathbf{r},t)$$

$$k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f$$

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau' \longrightarrow \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{\mathscr{r}} \, da' \longrightarrow \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{\mathscr{r}} \, dl' \longrightarrow \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i}$$

$$\mathbf{E} = -\vec{\nabla} V$$

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} \qquad \& \qquad \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0} \sigma$$

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1, j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i) \longrightarrow \frac{1}{2} \int \rho V \, d\tau$$

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$$

$$C \equiv \frac{Q}{\Delta V}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

$$V(\mathbf{r}) = V_{\text{monopole}}(\mathbf{r}) + V_{\text{dipole}}(\mathbf{r}) + \cdots$$

$$V_{\text{monopole}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

$$V_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \, \hat{\mathbf{r}} + \sin\theta \, \hat{\theta}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left[3(\mathbf{p} \cdot \hat{\mathbf{r}}) \, \hat{\mathbf{r}} - \mathbf{p} \right]$$

$$Q = \sum_{i=1}^{n} q_i \longrightarrow \int \rho(\mathbf{r}') \, d\tau'$$

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}_i \longrightarrow \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau'$$

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$\rho = \rho_b + \rho_f$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{D} = \rho_f \longrightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \text{(linear dielectrics)}$$

$$\mathbf{D} = \epsilon_0 (1 + \chi_e)\mathbf{E} \equiv \epsilon \mathbf{E} \quad \text{(linear dielectrics)}$$

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

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$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

$$H = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \longleftrightarrow \quad \oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\begin{aligned} \int_1^1 P_l(x) P_{l'}(x)\, dx &= \int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta \\ &= \begin{cases} 0, & \text{if } l' \neq l \\ \frac{2}{2l+1}, & \text{if } l' = l. \end{cases} \end{aligned}$$

$$\int_0^a \sin(n\pi y/a) \sin(n'\pi y a)\, dy = \begin{cases} 0, & \text{if } n' \neq n. \\ \frac{a}{2}, & \text{if } n' = n. \end{cases}$$