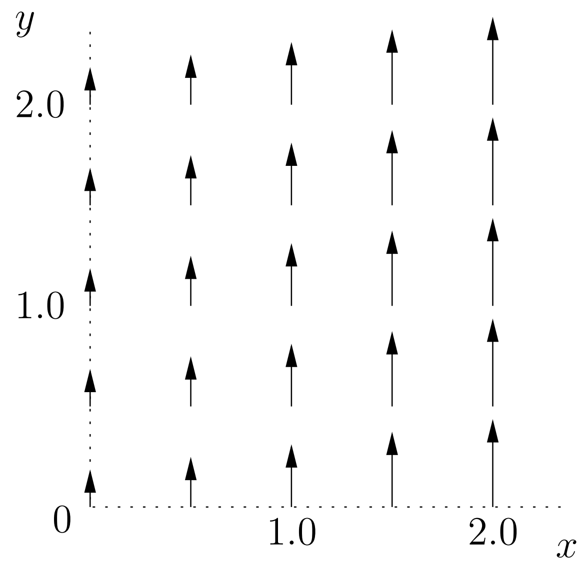
PHYS 333 — Final Exam
Friday December 13, 2013

Name:______________________

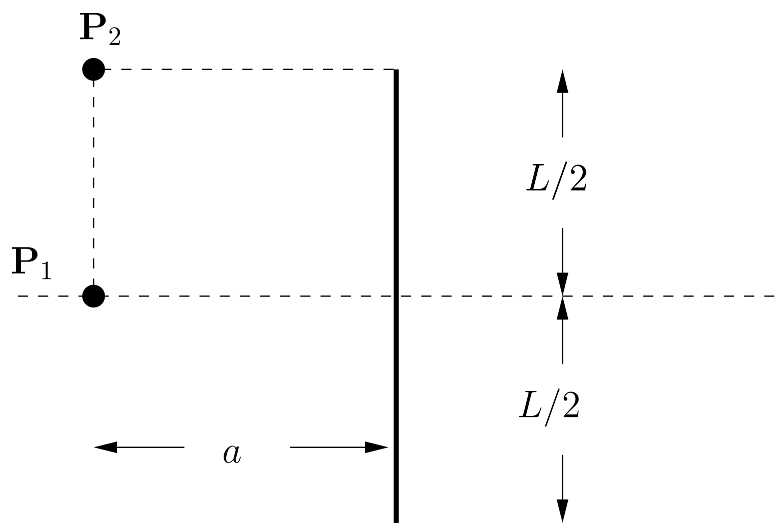
1. Consider the illustrated vector field. Assume that the field is the same in all planes parallel to the illustrated plane, i.e., the field at a point doesn't change as you move into or out of the page.



   (a) Is the divergence of the field at the center of the illustrated region ($x = 1$, $y = 1$) positive, negative, or zero? Explain your reasoning.

   (b) Is the z-component of the curl of the field at the center of the illustrated region ($x = 1$, $y = 1$) zero or non-zero? Explain your reasoning.

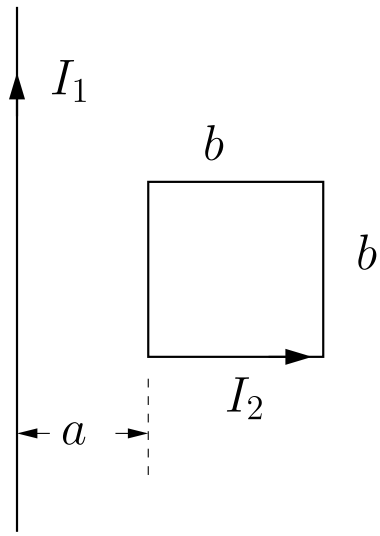
2. Consider the illustrated line of charge with length L and linear charge density λ.



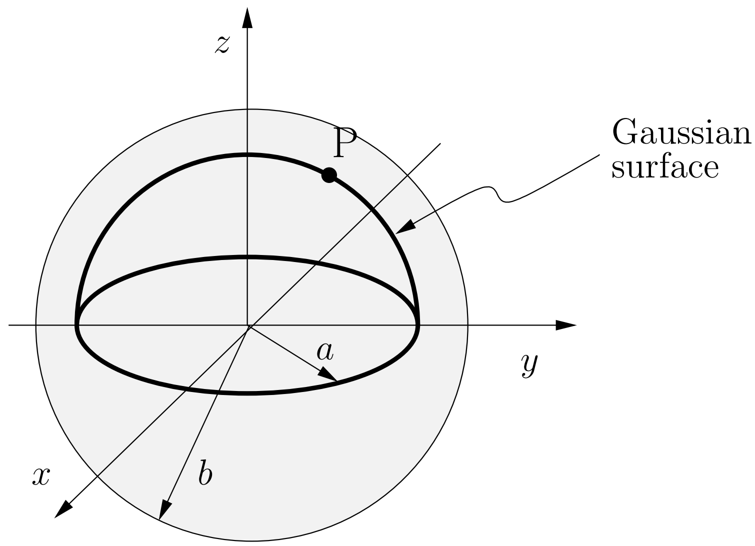
(a) Determine an expression for the electric field at point $\mathbf{P}_1$. You may leave your answer in the form of an integral which could be evaluated by a computer or calculator.

(b) Determine an expression for the electric field at point $\mathbf{P}_2$. You may leave your answer in the form of an integral which could be evaluated by a computer or calculator.

3. Find the force on the square loop with sides of length b that placed as shown in the figure a distance a away from an infinite straight wire. The long wire carries a steady current I_1 and the square loop carries a steady current I_2.

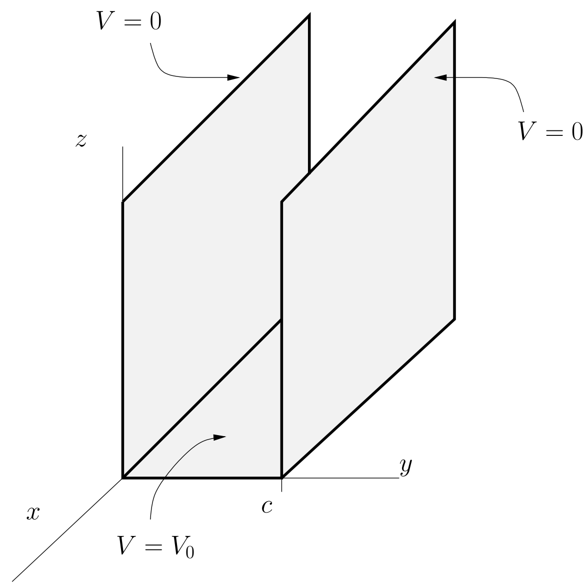


4. A sphere has charge uniformly distributed throughout its volume. The total charge is Q and the radius is b, and the sphere is centered on the origin. Determine the electric field vector $\mathbf{E}$ at the point P a distance a away from the origin, where $a < b$. **For full credit you must use the illustrated Gaussian surface in your solution.** This surface is a hemispherical dome of radius a with a flat bottom in the x-y plane.



5. *Begin* a solution of Laplace's equation, $\nabla^2 V = 0$, by separation of variables in spherical coordinates assuming there is no dependence on the azimuthal variable ϕ. You may stop as soon as you have determined the ordinary differential equations that must be solved for your product functions. You do not need to solve the differential equations.

6. Two semi-infinite grounded metal plates lie parallel to the x-z plane, one at $y = 0$, the other at $y = c$. These plates are joined to an infinite strip of width c lying in the x-y plane that is maintained at a constant potential V_0. (The strip is insulated from the grounded plates.)



The general solution to Laplace's equation in Cartesian coordinates for the potential can be written as

$$V(y, z) = \sum_n \left(A_n e^{k_n z} + B_n e^{-k_n z}\right) \left(C_n \sin k_n y + D_n \cos k_n y\right),$$

where the sum is over postive integers n. Use the boundary conditions to determine everything you can about the values of

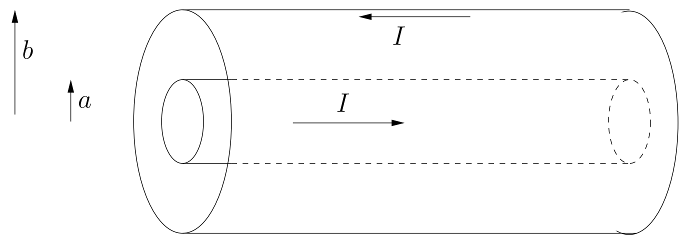
(a) k_n,

(b) A_n,

(c) D_n,

(d) the product $B_n C_n$.

7. A coaxial cable consists of two very long cylindrical conducting tubes, separated by a *linear* insulating material of magnetic susceptibility χ_m. A free current I flows down the the inner conductor and returns along the outer one; in each case the current distributes itself uniformly over the surface.



(a) Find **H** and **B** for all distances from the axis of the cable s, such that $s > a$

(b) Determine all bound currents.

8. Write down the electric and magnetic fields for a monochromatic plane wave propagating in free space with electric field amplitude E_0, angular frequency ω, phase angle zero, for the following cases.

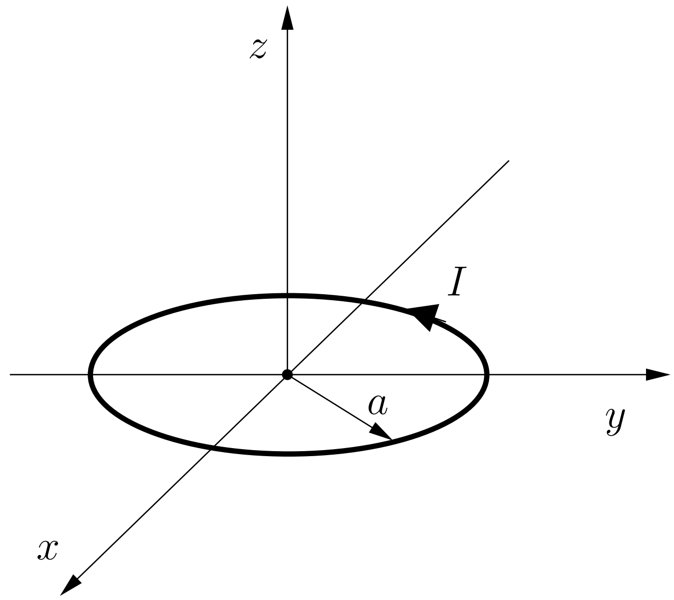
   (a) A wave traveling in the negative z direction and polarized in the y direction;

   (b) A wave traveling in the direction from the origin to the point $(1, 1, 0)$, with polarization parallel to the x-y plane.

   Your answer should be expressed in terms of E_0, ω, Cartesian coordinates, Cartesian unit vectors, and physical constants.

9. Determine the electric potential at the center of a uniformly charged spherical shell of radius R and total charge Q. **NOTE: This is a thin shell of charge; the charge is not distributed throughout the volume.** Use a point infinitely far away from the sphere as your reference point where $V = 0$.

10. Consider the illustrated loop of radius a in the x-y plane.



The magnetic field on the z-axis due to the illustrated current loop is

$$\mathbf{B} = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \,\hat{\mathbf{z}}$$

(a) Find a Taylor's series approximation for the field on the z-axis that is valid for points that are far from the origin. **Include the first two non-vanishing terms in your expansion.**

(b) Treat the current loop as a dipole, and determine the magnetic field due to the dipole for points on the z-axis. When is this expression valid?

Equations

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau' \longrightarrow \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da' \longrightarrow \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,dl' \longrightarrow \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

$$\mathbf{E} = -\vec{\nabla}V$$

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \qquad \& \qquad \frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0}\sigma$$

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1,j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \longrightarrow \frac{1}{2}\int \rho V\,d\tau$$

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\,d\tau$$

$$C \equiv \frac{Q}{\Delta V}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$V(\mathbf{r}) = V_{\text{monopole}}(\mathbf{r}) + V_{\text{dipole}}(\mathbf{r}) + \cdots$$

$$V_{\text{monopole}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

$$V_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right]$$

$$Q = \sum_{i=1}^{n} q_i \longrightarrow \int \rho(\mathbf{r}')\, d\tau'$$

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}_i \longrightarrow \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau'$$

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3} \left(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$\rho = \rho_b + \rho_f$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{D} = \rho_f \longrightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \quad \text{(linear dielectrics)}$$

$$\mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E} \equiv \epsilon \mathbf{E} \quad \text{(linear dielectrics)}$$

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

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$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \longleftrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da' \longleftrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$$\mathbf{I} = \lambda \mathbf{v}$$

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_\perp} \longrightarrow \sigma \mathbf{v}$$

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} \longrightarrow \rho \mathbf{v}$$

$$\mathbf{F} = q\,[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$

$$\mathbf{F}_{\text{mag}} = \int I(d\mathbf{l} \times \mathbf{B}) \longleftrightarrow \int (\mathbf{K} \times \mathbf{B})\, da \longleftrightarrow \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$$

$$\mathbf{m} = I \int d\mathbf{a}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}\right) = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$$

$$\mathbf{J} = \sigma \mathbf{E}$$

$$\mathcal{E} \equiv \oint \mathbf{f}_s \cdot d\mathbf{l} \longrightarrow \oint (\mathbf{f}_{\text{elec}} + \mathbf{f}_{\text{mag}}) \cdot d\mathbf{l}$$

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int \mathbf{B}\cdot d\mathbf{a} \longleftrightarrow \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\left(I_{\text{enc}} + \epsilon_0\frac{\partial}{\partial t}\int \mathbf{E}\cdot d\mathbf{a}\right) \longleftrightarrow \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{J}_d \equiv \epsilon_0\frac{\partial \mathrm{E}}{\partial t}$$

$$\Phi = LI \qquad \text{and} \qquad \Phi = MI$$

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$W = \frac{1}{2}LI^2$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\,d\tau = \frac{1}{2\mu_0}\int B^2 d\tau$$

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad \nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Traveling Plane Wave Solutions:

$$\mathbf{E}(\mathbf{r},t) = E_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\,\hat{\mathbf{n}} \qquad \mathbf{B}(\mathbf{r},t) = \frac{E_0}{c}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\,(\hat{\mathbf{k}}\times\hat{\mathbf{n}})$$

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi ya)\,dy = \begin{cases} 0, & \text{if } n' \neq n. \\ \frac{a}{2}, & \text{if } n' = n. \end{cases}$$

$$\mathbf{J}_b = \nabla\times\mathbf{M}$$

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

$$H = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$$