Problem F

Consider scattering a plane wave off a finite constant spherical potential well

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a. \end{cases}$$

using the method of partial waves and phase shifts.

(a) Show that the general interior solution is of the form

$$\psi = \sum_{l=0}^{\infty} A_l j_l(k_0 r) P_l(\cos\theta)$$

and derive an expression for k_0.

(b) The general exterior solution can be written as

$$\psi = \sum_{l=0}^{\infty} B_l \left[e^{i2\delta_l} h_l^{(1)}(kr) + h_l^{(2)}(kr) \right] P_l(\cos\theta).$$

Simplify the expression for the $l = 0$ mode for both the exterior solution and interior solution (i.e. get things in terms of trig functions and exponentials).

(c) For S-wave scattering, i.e., considering only $l = 0$, apply continuity of ψ and $\partial\psi/\partial r$ at the boundary $r = a$ and use the two equations to show

$$\tan(ka + \delta_0) = \frac{ka}{k_0 a} \tan(k_0 a)$$

(d) In the low energy limit, $ka \ll 1$ and the right hand side is small. Show that in this limit

$$\sigma = 4\pi a^2 \left(\frac{\tan(k_0 a)}{k_0 a} - 1 \right)^2$$

and that

$$k_0 a = \sqrt{\frac{2mV_0 a^2}{\hbar^2}}$$