Pulling on a Membrane

Let me fill in the details of the calculation we were doing in class today (Friday, April 11). We had motivated the free energy

$$G_{\text{tot}} = 12\pi K_b + \pi K_b \frac{L}{r} + G_{\text{stretch}} - \Delta p\left(\frac{4}{3}\pi R^3 + \pi r^2 L\right)$$

I am leaving out the G_{load} contribution. Instead, we'll calculate how G_{tot} depends on L and then take

$$f = \frac{dG_{\text{tot}}}{dL}$$

at the end of the calculation. And recall that we evaluated derivatives of the stretch free energy by using the area

$$a = 4\pi R^2 + 2\pi r L + \pi r^2$$

The last part is the area of the hemispherical cap minus the area of the circle cut out of the vesicle. Now we can evaluate

$$\frac{\partial G_{\text{stretch}}}{\partial R} = \underbrace{K_a\left(\frac{a - a_0}{a_0}\right)}_{=\tau}\frac{\partial a}{\partial R} = \tau\, 8\pi R$$

and similarly

$$\frac{\partial G_{\text{stretch}}}{\partial r} = \tau\frac{\partial a}{\partial r} = \tau 2\pi L + 2\pi r \simeq \tau 2\pi L, \qquad \frac{\partial G_{\text{stretch}}}{\partial L} = \tau\frac{\partial a}{\partial L} = \tau 2\pi r.$$

Fine, now take $\partial G_{\text{tot}}/\partial R = 0$ to get

$$8\pi\tau R - 4\pi\Delta p R^2 = 0 \qquad \Rightarrow \qquad \Delta p = 2\tau/R$$

which is the Laplace-Young relation. We will use this to eliminate Δp in favor of τ. Next we take $\partial G_{\text{tot}}/\partial r = 0$ to get

$$-\pi K_b\frac{L}{r^2} + 2\pi\tau L - 2\pi\Delta p r L = 0 \qquad \Rightarrow \qquad -\pi K_b\frac{L}{r^2} + 2\pi\tau L - 4\pi\tau L\frac{r}{R} = 0$$

Since $r \ll R$, the last term is negligible. Solving the resulting equation for r gives

$$r^2 = \frac{K_b}{2\tau} \qquad \Rightarrow \qquad r = \sqrt{\frac{K_b}{2\tau}} \quad \checkmark$$

Now we evaluate

$$\frac{\partial G_{\text{tot}}}{\partial L} = \frac{\pi K_b}{r} + 2\pi\tau r - \underbrace{\pi\Delta p r^2}_{=\pi K_b/R}$$

We can drop the last term since $R \gg r$. Subbing in the value for r then gives

$$f = \frac{\partial G_{\text{tot}}}{\partial L} = \pi\sqrt{2K_b\tau} + \pi\sqrt{2K_b\tau} = 2\pi\sqrt{2K_b\tau} \quad \checkmark$$

So this force is independent of L, which means it is a lousy way to pull a vesicle around! Instead, the calculation suggests that pulling with this force would just extend the cylinder ever farther and farther out.