Probability: Discrete Outcomes

General

- Discrete outcomes:
  \[ j = 1, 2, 3, \ldots \]

- Probability of outcome \( j \) : \( 0 \leq p_j \leq 1 \).

- Intuition: fraction of times various outcomes occur. With \( N \) trials and \( n_j \) occurrences of outcome \( j \),
  \[ p_j \approx \frac{n_j}{N}. \]

- After \( N \) trials, expect
  \[ n_j = p_jN \]
  occurrences of outcome \( j \).

Example

Random integers between 1 and 10.

- Outcomes
  \[ j = 1, 2, \ldots, 10. \]

- Probabilities (uniform case):
  \[ p_j = \frac{1}{10}. \]

- After 100 random numbers are generated, expect 10 occurrences of each outcome.

- Actual set of 100 trials:
  \[ n_1 = 12, \ n_2 = 13, \ldots \]
**Probability: Discrete Outcomes**

### General

- **Normalization:**
  \[ \sum_j p_j = 1. \]

- **Mean (expectation value):**
  \[ \langle j \rangle := \sum_j j p_j. \]

- **Standard deviation (uncertainty):**
  \[ \Delta j := \sqrt{\sum_j (j - \langle j \rangle)^2 p_j} = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}. \]

### Example

Random integers between 1 and 10.

- **Normalization:**
  \[ \sum_j p_j = \sum_{j=1}^{10} \frac{1}{10} = 1. \]

- **Mean:**
  \[ \langle j \rangle = \sum_j j p_j = 5.5 \]

- **Standard deviation:**
  \[ \langle j^2 \rangle = \sum_j j^2 p_j = 38.5 \]
  \[ \Delta j := \sqrt{38.5 - (5.5)^2} = 2.87. \]
Probability: Continuous Outcomes

General

- Real number outcomes, $x$.
- Probability density: $p(x) \geq 0$.
- Intuition: probability that outcome is in the range $x$ to $x + dx$:
  
  $$p(x) \, dx$$

- Probability that outcome is in the range $a \leq x \leq b$:

  $$Pr(a \leq x \leq b) = \int_{a}^{b} p(x) \, dx$$

Example

Random real numbers between 0 and 1.

- Uniform distribution:

  $$p(x)$$

- Gaussian distribution:
Probability: Continuous Outcomes

Probability is equivalent to area under probability density curve.

Probability and Area

Probability that outcome is in the range $a \leq x \leq b$:

$$\Pr(a \leq x \leq b) = \int_a^b p(x) \, dx.$$ 

Example

Gaussian distribution:
Probability: Continuous Outcomes

General

► Normalization:

\[ \int_{-\infty}^{\infty} p(x) \, dx = 1 \]

► Mean (expectation value):

\[ \langle x \rangle := \int_{-\infty}^{\infty} x \, p(x) \, dx \]

► Standard deviation (uncertainty):

\[ (\Delta x)^2 := \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 \, p(x) \, dx \]

\[ \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \]

Example

Uniformly distributed random real numbers between 0 and 1. Constant probability density \( A \).

► Normalization:

\[ \int_{-\infty}^{\infty} p(x) \, dx = 1 \Rightarrow A = 1. \]

► Mean:

\[ \langle x \rangle = \int_{-\infty}^{\infty} x \, p(x) \, dx = \frac{1}{2}. \]

► Standard deviation:

\[ \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \, p(x) \, dx = \frac{1}{3} \]

\[ \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{12}. \]