Suppose that you are given an ensemble of spin-1/2 particles all in the state $|\Psi\rangle$ and each is subjected to an $S_z\hbar$ measurement.

$|\Psi\rangle |\Psi\rangle |\Psi\rangle |\Psi\rangle$  
$\cdots \cdots \cdots$  
$S_n = +\frac{\hbar}{2}$  
$S_n = -\frac{\hbar}{2}$

Rather than record the individual outcomes, you can only get the average of all of the outcomes. Call the average $\overline{S_n}$. This is similar to the situation in NMR. There are many molecules:

$\begin{align*}
\text{Cl} \\
\text{Cl} - \text{C} - \text{H} \\
\text{Cl} \\
\text{Cl} - \text{C} - \text{H} \\
\text{Cl} \\
\text{Cl}
\end{align*}$

and the spectrometer can only get average measurements over the entire ensemble. In fact, it can measure $\overline{S_x}, \overline{S_y}$.

Can we easily predict what we will get? Again, there is some statistics involved but if the number of particles in the ensemble is very large, we expect that the expectation value $\langle S_n \rangle$ is close to the average that we get for the ensemble $\overline{S_n}$. So the issue is to be able to calculate $\langle S_n \rangle$ easily.
\[
\langle S_n \rangle = \frac{1}{2} \text{ Prob } (S_n = \frac{\hbar}{2}) + (-\frac{1}{2}) \text{ Prob } (S_n = -\frac{\hbar}{2})
\]

\[
= \frac{1}{2} \left\{ \left| \langle \hat{n} | \Psi \rangle \right|^2 - \left| \langle -\hat{n} | \Psi \rangle \right|^2 \right\}
\]

\[
= \frac{1}{2} \left\{ \langle \hat{\Psi} | \hat{n} \hat{\Psi} \rangle \ast \langle \hat{n} | \Psi \rangle - \langle -\hat{n} | \Psi \rangle \ast \langle -\hat{n} | \Psi \rangle \right\}
\]

\[
= \frac{1}{2} \left\{ \langle \Psi | \hat{n} \hat{\Psi} \rangle \ast \langle \hat{n} | \Psi \rangle - \langle \Psi | -\hat{n} \hat{\Psi} \rangle \ast \langle -\hat{n} | \Psi \rangle \right\}
\]

Since \( \langle \Psi | \Psi \rangle \ast = \langle \Psi | \Psi \rangle \), then it follows that

If every particle is in the state \( |\Psi \rangle \) then

\[
\langle S_n \rangle = \langle \Psi | \hat{S}_n | \Psi \rangle
\]

where

\[
\hat{S}_n = \frac{\hbar}{2} \left\{ \left| \hat{A} \hat{X} + \hat{n} \right| - \left| \hat{-A} \hat{X} - \hat{n} \right| \right\}
\]

The operator \( \hat{S}_n \) corresponds to measurements of the spin component along \( \hat{n} \) in the sense above. This operator is called an observable corresponding to \( S_n \). Since all the measurements for spin-\( \frac{1}{2} \) systems are SG measurements, we can always find a suitable observable for each.

Note that for spin-\( \frac{1}{2} \) components:

\[
\hat{S}_n = \frac{1}{2} \hat{S}_n
\]
Example: Suppose that an ensemble of spin-half particles are all in the $\hat{z}$ state and are subjected to $SG\hat{y}$.

Determine $\langle S_y \rangle$.

\[ \langle S_y \rangle = \langle \Phi | \hat{S}_y | \Phi \rangle. \]

But $\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Thus

\[ \langle S_y \rangle = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0 \]

as one may have expected.

Hermitian operators.

Note that any observable satisfies:

\[ \hat{S}_n^+ = \hat{S}_n \]

This is called Hermitian. In general, every observable is Hermitian and every Hermitian operator can be cast as an observable. One can describe measurements via observables (equivalent to Hermitian operators).
Reconstructing measurements from observables.

Suppose that $\hat{A}$ is some observable operator. Can we reconstruct the usual language of measurements in terms of states that give definite outcomes? That is, for a spin-$\frac{1}{2}$ system can we find

<table>
<thead>
<tr>
<th>outcomes</th>
<th>states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$14_1\rangle$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$14_2\rangle$</td>
</tr>
</tbody>
</table>

so that, when

\[
14_1\rangle \quad \xrightarrow{\hat{A}} \quad a_1 \quad \text{certainty} \\
\text{apparatus} \quad \text{measuring} \quad \hat{A} \\
\ldots \quad a_2 \quad \text{never}.
\]

etc...

Let's consider an example of measuring the $\hat{Z}$ component of spin. The observable is

\[
\hat{S}_z = \frac{\hbar}{2} \left( \hat{X}\hat{Z} + \hat{Z}\hat{X} \right) - 1\hat{\hat{Z}} X - \hat{\hat{Z}} I
\]

and we know that the states we want are

\[
1\hat{\hat{Z}} \rangle \rightarrow \text{always gives } S_z = +\frac{\hbar}{2} \\
1\hat{-\hat{Z}} \rangle \rightarrow \text{always } S_z = -\frac{\hbar}{2}.
\]

How can we extract these from $\hat{S}_z$?
Aside from reading off from the explicit expression we could try:

\[ \hat{S}_z \, |\pm \rangle = \pm \frac{\hbar}{2} \, |\pm \rangle \]
\[ \hat{S}_z \, |\mp \rangle = \mp \frac{\hbar}{2} \, |\mp \rangle \].

These both have the form of an eigenvalue equation

\[ \hat{S}_z |\Psi \rangle = \lambda |\Psi \rangle \]

for some real \( \lambda \). You could show that no other states obey this.

Thus the scheme for a general observable is: Solve the eigenvalue equation:

\[ \hat{A} |\Psi_i \rangle = \alpha_i |\Psi_i \rangle \]

Then \( |\Psi_i \rangle \) gives outcome \( \alpha_i \) with certainty. The state \( |\Psi_i \rangle \) is called an eigenstate and the associated \( \alpha_i \) an eigenvalue. Thus we get the following for a general input state: \( |\Psi \rangle \):

<table>
<thead>
<tr>
<th>outcome</th>
<th>associated state</th>
<th>probability of outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td>(</td>
<td>\Psi_1 \rangle )</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>(</td>
<td>\Psi_2 \rangle )</td>
</tr>
</tbody>
</table>

For this to be reasonable we need the following results:

Theorem: \( \hat{A} \) Hermitian \( \Rightarrow \) eigenvalues of \( \hat{A} \) are real.

Theorem: \( \hat{A} \) Hermitian and \( \alpha_1, \alpha_2 \) distinct eigenvalues
\( \Rightarrow \) associated eigenstates are orthogonal.