Energy eigenstates + Time evolution.

So far we have discussed energy eigenstates, $| \psi_n \rangle$ which satisfy

$$\hat{H} | \psi_n \rangle = E_n | \psi_n \rangle$$

In terms of wavefunctions,

$$\left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right\} \psi_n(x) = E_n \psi_n(x).$$

We also noted that the most general state can be expressed as a superposition

$$| \Psi \rangle = \sum_{n=1}^{\infty} C_n(t) | \psi_n \rangle \quad \rightarrow \quad \Psi(x,t) = \sum_{n=1}^{\infty} C_n(t) \psi_n(x)$$

Then we saw that the system evolves via:

$$| \Psi(0) \rangle \rightarrow | \Psi(t) \rangle = e^{-iHt/\hbar} | \Psi(0) \rangle.$$ 

and that

$$e^{-iHt/\hbar} | \psi_n \rangle = e^{-iE_nt/\hbar} | \psi_n \rangle.$$ 

Finally suppose that a system is in the state

$$| \Psi(0) \rangle = \sum_{n=1}^{\infty} C_n | \psi_n \rangle$$

It follows that

$$| \Psi(t) \rangle = \sum_{n=1}^{\infty} C_n e^{-iE_nt/\hbar} | \psi_n \rangle \quad \rightarrow \quad \Psi(x,t) = \sum_{n=1}^{\infty} C_n e^{-iE_nt/\hbar} \psi_n(x).$$
Example: Suppose $|\Psi(0)\rangle = 1_{n}\rangle$ for some $n$. Then

$$|\Psi(t)\rangle = e^{-i\Delta t/t} |\Psi_{n}\rangle$$

or, in terms of wavefunctions,

$$\Psi(x,t) = e^{-i\Delta t/t} \Psi_{n}(x)$$

Thus the position probability density is

$$|\Psi(x,t)|^2 = |\Psi_{n}(x)|^2$$

and is independent of $t$. So here $e^{-i\Delta t/t}$ is a global phase
and is irrelevant for physical predictions.

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Position Wavefunctions. Townsend 6.1

We have seen that every state of a quantum system can be represented as a superposition of energy eigenstates:

$$|\Psi\rangle = \sum c_{n} |\Psi_{n}\rangle$$

and all you need to describe it are

- eigenstates $|\Psi_{n}\rangle$
- coefficients $c_{n}$.

Recall that $c_{n} = \langle \Psi_{n}|\Psi\rangle$, gives a method for computing $c_{n}$.

But we have seen that wavefunctions can also be used to describe the state of a quantum system. How can we extract these from the state $|\Psi\rangle$?

We start by introducing "position eigenstates"

$$|x\rangle \quad \text{particle will be located at position } x \text{ with certainty}$$

for all real $x$.

Such eigenstates, it turns out do not truly belong in the set of quantum states, but they are useful in calculations. That is why we should be careful about saying, a particle is in state $|x\rangle$ or $|x,\rangle$ etc....
We can however, construct superpositions of these states:

\[ |\Psi(t)\rangle = \int_{-\infty}^{\infty} \Psi(x,t) |x\rangle. \]

Again, you may suspect that you will actually have to evaluate such an integral in closed form. This is not the case. Rather you should think of the various \(|x\rangle\) kets as distinct basis vectors, the \(\Psi(x,t)\) as components and \(\int_{-\infty}^{\infty} dx\) as a sum over all possible basis vectors.

Before doing any calculations, you will need the notion of an inner-product between basis vectors \(\langle x | x' \rangle\). Clearly if \(x = x'\) then \(\langle x | x' \rangle = 0\) but what if \(x = x'\)? We will insist on the following rule:

\[ \langle x | x' \rangle = \delta(x-x') \]

where \(\delta(x-x')\) is the Dirac delta function. This function is defined as

\[ \delta(x-x') = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases} \]

and such that

\[ \int_{-\infty}^{\infty} dx \, f(x) \delta(x-x') = f(x') \quad \text{or} \quad \int_{-\infty}^{\infty} dx \, f(x) \delta(x-x') = f(x') \]

In particular

\[ \int_{-\infty}^{\infty} dx \, \delta(x-x') = 1. \quad \text{(unit area under } \delta(x-x')\text{)} \]

You can picture the Dirac delta function as a sequence of constant functions:
Now take the limit as $\epsilon \to 0$ and you get $\delta(x-x')$.

Note that this gives

$$
\langle x | \Psi(t) \rangle = \langle x | \int_{-\infty}^{\infty} dx' \, \Psi(x',t) | x' \rangle \\
= \int_{-\infty}^{\infty} dx' \, \Psi(x',t) \langle x | x' \rangle \\
= \Psi(x,t)
$$

So we get an analogous rule for the position-basis components

$$
\Psi(x,t) = \langle x | \Psi(t) \rangle.
$$

Furthermore note that the usual rules for manipulating bras and kets apply. Thus:

$$
(\Psi(x,t))^\dagger = (\langle x | \Psi(t) \rangle)^\dagger = \langle \Psi(t) | x \rangle \\
= \Psi^*(x,t)
$$

and thus

$$
\Psi^*(x,t) = \langle \Psi(t) | x \rangle.
$$
Lastly there is a completeness relation for $|x\rangle$ basis:

$$\int_{-\infty}^{\infty} dx \, |x\rangle \langle x| = \mathbb{1}$$

Again this will be used as a calculating tool, and you are not expected to explicitly integrate.

**Example:** These can be used to compute inner products. So

$$\langle \Phi | \Psi \rangle = \langle \Phi | \mathbb{1} | \Psi \rangle = \langle \Phi | \int_{-\infty}^{\infty} dx \, |x\rangle \langle x| | \Psi \rangle = \int_{-\infty}^{\infty} dx \, \Phi^*(x) \Psi(x)$$

Here one never actually evaluated $\int_{-\infty}^{\infty} dx \, |x\rangle \langle x| | \Psi \rangle$.

**Position Operator.**

One can measure the position of the particle. Thus there must be an observable that corresponds to it. In fact we use

$$\hat{x} |x\rangle = x |x\rangle$$

for all real $x$. This is the eigenvalue equation for $\hat{x}$. Now suppose that you want to determine

$$| \Phi \rangle := \hat{x} | \Psi \rangle$$

for an arbitrary state $| \Psi \rangle$. 
We show how this can be viewed in terms of wavefunctions:

\[ \hat{\Phi}(x,t) = \langle x | \hat{\Phi}(t) \rangle. \]

\[ = \langle x | \hat{x} | \Psi \rangle. \]

\[ = \langle x | \hat{x} | \hat{I} | \Psi \rangle. \]

\[ = \int_{-\infty}^{\infty} dx' \langle x | x' \rangle \langle x' | \Psi \rangle. \]

\[ = \int_{-\infty}^{\infty} dx' \langle x | x' \rangle \Psi(x',t). \]

\[ = \int_{-\infty}^{\infty} dx' \Psi(x',t) \langle x | x' \rangle \frac{\delta(x-x')}{\delta(x-x')} \]

\[ = \int_{-\infty}^{\infty} dx' \Psi(x',t) \delta(x-x') = x \Psi(x,t). \]

Thus

\[ |\Phi\rangle = \hat{x} |\Psi\rangle \quad \rightarrow \quad \Phi(x,t) = x \Psi(x,t). \]

A similar argument results in:

\[ \langle \Phi | \hat{x} | \Phi \rangle = \int_{-\infty}^{\infty} dx \Phi^*(x,t) x \Phi(x,t) \]

and the expectation value of \( \hat{x} \) is:

\[ \langle x \rangle = \langle \Phi(t) | \hat{x} | \Phi(t) \rangle. \]