Free particle.

We consider a free particle whose initial position space wavefunction is the Gaussian:

\[
\Psi(x,0) = \frac{1}{\pi^{1/4}/\sqrt{a}} \, e^{-|x-x_0|^2/2a^2}
\]

Note that this gives:

\[
\langle x \rangle = x_0 \\
\Delta x = \frac{\hbar}{\sqrt{2}}
\]

Now the corresponding momentum space wavefunction is:

\[
\widetilde{\Psi}(p,0) = \frac{1}{\sqrt{2\pi}\hbar} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,0) dx
\]

and the Gaussian integral gives:

\[
\widetilde{\Psi}(p,0) = \frac{1}{\pi^{1/4}/\sqrt{a}} \, e^{-ipx_0/\hbar} \, e^{-p^2a^2/2\hbar^2}
\]

from which one can compute:

\[
\langle p \rangle = 0 \\
\Delta p = \frac{\hbar}{\sqrt{2a}}
\]

momentum expectation value

momentum uncertainty

Note that there is a trade off between position + momentum uncertainties.
In fact this gives $\Delta x \Delta p = \hbar/2$ which is the best that can be managed according to the uncertainty principle.

Time evolution  
*Townsend* p.163

The general rule for time evolution is:

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\Psi(0)\rangle$$

where

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

What concerns us will be $\Psi(x,t)$. 
Now
\[ \Psi(x,t) = \langle x | \Psi(t) \rangle \]
\[ = \langle x | e^{-i\hat{p}^2 / 2m t} | \Psi(0) \rangle \]

How can one evaluate this? We can use the fact that for a momentum eigenstate, \( |p \rangle \)
\[ e^{-i\hat{p}^2 t / 2mt} |p \rangle = e^{-i\hat{p}^2 t / 2mt} |p \rangle \]
\[ \uparrow \quad \text{operator} \]
\[ \uparrow \quad \text{real number} \]

Thus try to insert \( \int_{-\infty}^{\infty} dp |p\rangle \langle p| \) in the above expression.
\[ \Psi(x,t) = \langle x | e^{-i\hat{p}^2 / 2mt} \int_{-\infty}^{\infty} dp |p\rangle \langle p| \Psi(0) \rangle \]
\[ = \int_{-\infty}^{\infty} dp \ e^{-i\hat{p}^2 / 2mt} \langle x|p\rangle \langle p| \Psi(0) \rangle \]

\[ \Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar m t}} \int_{-\infty}^{\infty} dp \ e^{-i\hat{p}^2 / 2mt} \ e^{i\hbar/p} \tilde{\Psi}(p,0) \]

and using the earlier expression for \( \tilde{\Psi}(p,0) \) we can now hopefully evaluate \( \Psi(x,t) \). When \( \tilde{\Psi}(p,0) \) corresponds to a Gaussian wavepacket, the integral can be evaluated explicitly.
Thus
\[ \Psi(x,0) = \frac{1}{\pi^{1/4}a} \ e^{- (x-x_0)^2 / 2a^2} \]
gives
\[ \Psi(x,t) = \frac{1}{\pi^{1/4}} \ \sqrt{\frac{am}{a^2m + it\hbar}} \ \exp \left\{ - \frac{m(x-x_0)^2}{2(a^2m + it\hbar)} \right\} \]
\[ = \frac{1}{\pi^{1/4}} \ \sqrt{\frac{1}{a + it\hbar/m}} \ \exp \left\{ - \frac{(x-x_0)^2}{2(a^2 + it\hbar/m)} \right\} \]
and some elementary algebra gives:
\[ \Psi(x,t) = \frac{1}{\pi^{1/4}} \ \sqrt{\frac{a^2m^2 + t^2\hbar^2}{a^2m^2 + t^2\hbar^2}} \ \exp \left\{ - \frac{(x-x_0)^2 a^2}{2(a^2 + t^2\hbar/m^2)} \right\} \ \exp \left\{ \frac{i(x-x_0)^2 t^2\hbar}{2(a^2 + t^2\hbar/m^2)} \right\} \]

Note that this is again a Gaussian centered at \( x_0 \). The position probability density is:
\[ |\Psi(x,t)|^2 = \frac{1}{\pi^{1/4}} \ \sqrt{\frac{am}{a^2m^2 + t^2\hbar^2}} \ \exp \left\{ - \frac{(x-x_0)^2 a^2}{a^2m^2 + t^2\hbar^2} \right\} \]

It follows that the expectation value of position is:
\[ \langle x \rangle = x_0 \]
and the uncertainty
\[ \Delta x = \frac{\hbar}{\sqrt{2}} \ \sqrt{\frac{a^2m^2 + t^2\hbar^2}{a^2m^2}} = \frac{a}{\sqrt{2}} \ \sqrt{1 + \frac{t^2\hbar^2}{a^2m^2}} \]
and note that this increases as \( t \) increases.
Exercise: Suppose that we can locate a typical proton (mass 7010 g) at t = 0 with the precision of the width of an atom. So

\[ \Delta x \text{ at } t = 0 = 10^{-10} \text{ m}. \]

Determine the time at which \( \Delta x = 10^{-6} \text{ m}. \)

Answer:

\[ \Delta x_{\text{old}} = 10^{-10} \text{ m} = \frac{a}{\sqrt{2}} \]
\[ \Delta x_{\text{new}} = 10^{-6} \text{ m} = \frac{a}{\sqrt{2}} \sqrt{1 + \frac{t^2 \hbar^2}{a^4 m^2}} \]

\[ \Rightarrow a = \sqrt{2} \times 10^{-10} \text{ and} \]
\[ \frac{\Delta x_{\text{new}}}{\Delta x_{\text{old}}} = \sqrt{1 + \frac{t^2 \hbar^2}{a^4 m^2}} = 10^9 \]

\[ \Rightarrow 1 + \frac{t^2 \hbar^2}{a^4 m^2} = 10^8 \]

\[ \Rightarrow \frac{t^2 \hbar^2}{a^4 m^2} = 10^8 \Rightarrow t^2 = \frac{m^2}{\hbar^2} \times 10^8 \]

\[ \Rightarrow t = \frac{M}{\hbar} \times 10^5 \]

\[ = \frac{M}{\hbar} \times 2 \times 10^{-20} \times 10^4 \]

\[ = \frac{2M}{\hbar} \times 10^{-16} \]

\[ = 2m \cdot \frac{10^{-16}}{\frac{2\pi}{\hbar}} = 4\pi m \times 10^{-6} \]

\[ = 6.63 \times 10^{-34}. \]

\[ \Rightarrow t = 1.89 \times 10^{28} \text{ m} \]

For a proton, \( t = 133 \times 10^{28} \text{ s} \).