Three dimensional harmonic oscillator.

Recall that the energy eigenstates for a three dimensional harmonic oscillator are:

\[ \Psi(x, y, z) = \Psi_{n_x}(x) \Psi_{n_y}(y) \Psi_{n_z}(z) \]

where \( \Psi_{n_x}(x) \) is the position-space wavefunction for the one dimensional oscillator energy eigenstate \( |n_x\rangle \). Note that the state requires three labels \( n_x, n_y, n_z \) and it can be denoted \( |n_x, n_y, n_z\rangle \).

The corresponding energy eigenvalues are:

\[ E = \hbar \omega_x (n_x + \frac{1}{2}) + \hbar \omega_y (n_y + \frac{1}{2}) + \hbar \omega_z (n_z + \frac{1}{2}) \]

An interesting special case is the isotropic harmonic oscillator, for which \( \omega_x = \omega_y = \omega_z = \omega \). In this case the energy eigenvalues are

\[ E = \hbar \omega (n_x + n_y + n_z + \frac{3}{2}). \]

You can easily show that there are several different eigenstates sharing the same eigenvalue. These arise from different combinations of \( n_x, n_y, n_z \).

| \( \frac{3\hbar \omega}{2} \) | 1 |
| \( \frac{5\hbar \omega}{2} \) | 3 |
| \( \frac{7\hbar \omega}{2} \) | 6 |
You may ask where these numbers come from. The main clue will be that some of them look similar to the kinds of numbers that occur in atomic energy states: s→ 1, p→ 3, d→ 5, etc... or at least simple combinations of these. We shall encounter the deeper reason later; these have to do with the symmetric nature of the potential and the consequences of this for angular momentum.

Meanwhile, let's see that some of the states for an individual energy level are indeed different. Consider the second lowest level where one of nₓ, nᵧ, n₀ is 1 and the others are 0. Two states that give the same energy are:

\[ |100\rangle \rightarrow \Psi₄(x) \Psi₀(y) \Psi₀(z) \quad \text{and} \quad \Psi₀(x) \Psi₄(y) \Psi₀(z) \equiv |100\rangle. \]

We shall ignore the behavior as a function of z and plot these as functions of x and y. Note that

\[ \Psi₀(x) = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)^{1/4} e^{-\frac{mx^2}{2}} \]

\[ \Psi₄(x) = \left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)^{1/4} \sqrt{\frac{2mw}{\hbar}} x e^{-\frac{mx^2}{2}} \]

For plotting purposes it is convenient to note that \( \sqrt{\frac{\pi}{\sqrt{2}m\omega}} \) has units of length. Thus we can use a dimensionless quantity

\[ x' = \sqrt{\frac{m\omega}{\hbar}} x \]

The additional factor of \( \pi \) arises from the fact that the wavefunction also has units - in this case of \( (\text{length})^{-1/2} \) [Convince yourself of this].
Thus we can rescale the wave function as well:

\[ \Psi'_n(x') := \left( \frac{\pi k}{m \omega} \right)^{1/4} \Psi_n(x). \]

So we get rescaled wave functions:

\[ \Psi'_0(x') = e^{-x'^2 \pi/2} \]

\[ \Psi'_1(x') = \sqrt{2\pi} x' e^{-x'^2 \pi/2}. \]

You can check that these are properly normalized.

Now we can combine these and plot as a function of \( x' \).

First consider the states:

\[ \Psi'_0(x') \Psi'_1(y') \quad \text{vs} \quad \Psi'_1(x') \Psi'_0(y') \]

A plot of these shows that they are quite different. It may be clear that they are orthogonal.

Also consider the states

\[ \frac{1}{\sqrt{2}} \left\{ \Psi'_0(x') \Psi'_1(y') + \Psi'_1(x') \Psi'_0(y') \right\}. \]

These also have energy \( \frac{5\hbar \omega}{2} \) (once the \( \Psi'_0(x') \) part is included).

Look at the plots of these. Can you see what combination of position measurements could reveal a difference?
We will often encounter circumstances in which the interaction that the particle experiences depends only on the distance from some central point and not on the direction from this point.

**Example 1**: A point charged particle in a Coulomb potential:

\[ V = \frac{k q_1 q_2}{r} \]

**Example 2**: An isotropic harmonic oscillator

\[ V = \frac{1}{2} m \omega^2 r^2 \]
\[ = \frac{1}{2} m \omega^2 (x^2 + y^2 + z^2) \]

In these cases the potential only depends on the distance from the origin.

\[ r = \sqrt{x^2 + y^2 + z^2} \]

Generally we can consider a potential which only depends on \( r \)

\[ V = V(r) \]

does not depend on \( \theta, \phi \)

- In this case the physical situation is spherically symmetric.
Classical assessment.

In classical mechanics the net force is:

\[ \vec{F}_{\text{net}} = - \vec{\nabla} V(r) \]

Using a spherical co-ordinate system, one can show that \( \vec{F}_{\text{net}} \) is always directed radially inward or outward. If it had any component in a direction orthogonal to these then a symmetry argument would insist that it have the same components in all orthogonal directions. This one can envisage.

Now consider the torque about the central point.

\[ \vec{\tau}_{\text{net}} = \vec{r} \times \vec{F}_{\text{net}} = 0 \]

This must have some implications for angular momentum. Indeed,

\[ \frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}} = 0 \]

and this gives a very convenient result.

V is spherically symmetric \( \Rightarrow \vec{L} \) conserved
This is an example of a connection between symmetry and conservation laws. In physics we will often be able to exploit this.

**Quantum Treatment.**

We shall extend these ideas into the realm of quantum mechanics concentrating on angular momentum. The program is as follows:

1) Consider potentials which are spherically symmetric — that is, they are invariant under arbitrary rotations. Classical considerations suggest that angular momentum will be useful.

2) Define angular momentum operators, ascertain some of their simple properties.

3) Show that for spherically symmetric Hamiltonians, the angular momentum operators commute with the Hamiltonian. This means:

   - the angular momentum is conserved.
   - there are simultaneous eigenstates of energy and some angular momenta.

4) Find the eigenstates of angular momentum — use these to help construct the energy eigenstates.