Angular momentum in Quantum Mechanics: Central Potentials.

The general scheme for dealing with central potentials will be:

1) First consider classical description and rewrite the energy in terms of angular and radial momentum.

2) Define angular momentum operators in quantum mechanics and determine some of their properties.

3) Show that the Hamiltonian commutes with angular momentum operators.
   - conservation of angular momentum
   - one can always arrange for every energy eigenstate to also be eigenstates of some of the angular momenta

4) Rewrite the Hamiltonian in terms of angular momentum + radial momentum.

5) Find angular momentum eigenstates.

Classical treatment.

Consider a classical particle in a spherically symmetric potential. The total energy is

\[ E = \frac{1}{2m} (P_x^2 + P_y^2 + P_z^2) + V(r) \]

We note that there are two co-ordinate systems here: Cartesian for the momenta and spherical for the potential. Can we rewrite the momenta?
To get a grasp of this, consider a particle moving in the x-y plane.

The momentum can be expressed in terms of an orbital component, \( p_\phi \), and a radial component \( p_r \). You would expect that the orbital component is related to angular momentum and the radial component is independent of angular momentum.

Specifically, the meaning of these components comes from expressing \( \vec{p} \) in terms of unit vectors appropriate for spherical co-ordinates \( \hat{r}, \hat{\theta}, \hat{\phi} \).

Then for a particle in the x-y plane:

\[
\vec{p} = p_r \hat{r} + p_\phi \hat{\phi}
\]

and for a particle moving in three dimensions:

\[
\vec{p} = p_r \hat{r} + p_\phi \hat{\phi} + p_\theta \hat{\theta}
\]

How do we relate these to the Cartesian components?

\[
\vec{p} = p_x \hat{x} + p_y \hat{y} + p_z \hat{z}
\]

This requires a detailed treatment of the transformations between spherical and Cartesian co-ordinates.
Once this is done you can express $p_x, p_y, p_z$ in terms of $p_r, p_\theta, p_\phi$ and use these to express the energy in terms of $p_r, p_\theta, p_\phi$. In this way, you introduce the radial component of momentum.

The next step is to introduce angular momentum. The three components are:

\[
\begin{align*}
L_x &= y p_z - z p_y \\
L_y &= z p_x - x p_z \\
L_z &= x p_y - y p_x
\end{align*}
\]

These can also be written in terms of spherical co-ordinates. When all is done you find

\[
p_x^2 + p_y^2 + p_z^2 = p_r^2 + \frac{1}{r^2} \tilde{L}^2
\]

where

\[
\tilde{L}^2 = L_x^2 + L_y^2 + L_z^2
\]

Thus

\[
E = \frac{p_r^2}{2m} + \frac{1}{2mr^2} \tilde{L}^2 + V(r)
\]
We will see that the Hamiltonian in for the quantum system can be written in a similar way.

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To reformulate the Hamiltonian we will need angular momentum operators. These are:

\[
\hat{L}_x = \hat{y} \hat{p}_z - \hat{z} \hat{p}_y \\
\hat{L}_y = \hat{z} \hat{p}_x - \hat{x} \hat{p}_z \\
\hat{L}_z = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x
\]

x component

y component

z component.

Now, using the commutation rules for position + momentum you can deduce that

\[
[\hat{L}_x,\hat{L}_y] = i\hbar \hat{L}_z \\
[\hat{L}_y,\hat{L}_z] = i\hbar \hat{L}_x \\
[\hat{L}_z,\hat{L}_x] = i\hbar \hat{L}_y
\]

Question: What does this imply for measurements of angular momentum?

Can you find states which have definite values of all components of angular momentum?

Answer: Since \( \hat{L}_x, \hat{L}_y, \hat{L}_z \) do not commute, this is impossible. We can speak of states with definite \( x \) component but uncertain \( y \) component etc...
However, we can show that we can speak of total angular momentum + any single component. We define a magnitude of total angular momentum squared operator.

\[ L^2 = L_x^2 + L_y^2 + L_z^2 \]

and one can show

\[ [L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0 \]

So there are states of simultaneous magnitude squared and \( L_z \).

**Angular momentum in central potentials.**

How does this apply to particles in central potentials? We can show that if

\[ \hat{H} = \frac{1}{2m} (\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2) + V(r) \]

then

\[ [\hat{H}, \hat{L}_i] = 0 \quad i = x, y, z \]

and thus

\[ [\hat{H}, L^2] = 0 \]

Immediately Ehrenfest's theorem implies that for any state:

\[ \langle L_x \rangle, \langle L_y \rangle, \langle L_z \rangle \]

\[ \langle L^2 \rangle \]

are all constant. Thus angular momentum is conserved in quantum mechanics.
Additionally, it tells you that there are simultaneous eigenstates of \( \hat{H}, \hat{E}^2 \), and any single one of \( \hat{L}_x, \hat{L}_y, \hat{L}_z \). It is conventional to use \( \hat{L}_z \). So we will look for states

\[ |E, \Lambda, \lambda \rangle \]

such that

\[ \hat{H} |E, \Lambda, \lambda \rangle = E |E, \Lambda, \lambda \rangle \]
\[ \hat{L}_z^2 |E, \Lambda, \lambda \rangle = \Lambda |E, \Lambda, \lambda \rangle \]
\[ \hat{L}_z |E, \Lambda, \lambda \rangle = \lambda |E, \Lambda, \lambda \rangle \]

Re-expressing the Hamiltonian in terms of angular momentum operators

Just like the classical case, we will need to re-express the Hamiltonian in terms of angular momentum. In this case we expect the \( \hat{L}_z^2 \) operator to appear. But we also need a radial momentum operator. Classically it turns out that

\[ \hat{P}_r = \frac{x}{r} \hat{P}_x + \frac{y}{r} \hat{P}_y + \frac{z}{r} \hat{P}_z \]

where \( r = \sqrt{x^2 + y^2 + z^2} \). What operator corresponds to \( \hat{r} \)?

One may expect

\[ \hat{\frac{x}{r}} \hat{P}_x + \hat{\frac{y}{r}} \hat{P}_y + \hat{\frac{z}{r}} \hat{P}_z \]

but why not

\[ \hat{P}_x \hat{\frac{x}{r}} + \hat{P}_y \hat{\frac{y}{r}} + \hat{P}_z \hat{\frac{z}{r}} \]
These are different, e.g.

\[ \hat{p}_x \frac{\hat{x}}{\hat{r}} |\psi\rangle \longrightarrow -i\hbar \frac{\partial}{\partial x} \left[ \frac{x}{(x^2+y^2+z^2)^{3/2}} \psi(x,y,z) \right] \]

\[ = -i\hbar \frac{x}{(x^2+y^2+z^2)^{1/2}} \frac{\partial \psi}{\partial x} - i\hbar \frac{\psi}{(x^2+y^2+z^2)^{1/2}} \]

\[ + i\hbar \frac{x}{(x^2+y^2+z^2)^{3/2}} \psi. \]

\[ \leftarrow D \quad \hat{p}_x \frac{\hat{x}}{\hat{r}} |\psi\rangle \rightarrow -i\hbar \frac{1}{\hat{r}} |\psi\rangle + i\hbar \frac{\hat{x}^2}{\hat{r}^3} |\psi\rangle. \]

Thus

\[ \hat{p}_x \frac{\hat{x}}{\hat{r}} = \frac{\hat{x}^2}{\hat{r}^3} + i\hbar \frac{\hat{x}}{\hat{r}^3} + i\hbar \frac{\hat{x}}{\hat{r}} \frac{\partial}{\partial x} - i\hbar \frac{x}{\hat{r}^3} \frac{\partial}{\partial x} \]

So we really need

\[
\hat{p}_r := \frac{1}{2} \left( \frac{\hat{x}}{\hat{r}} \hat{p}_x + \hat{p}_x \frac{\hat{x}}{\hat{r}} + \frac{\hat{y}}{\hat{r}} \hat{p}_y + \hat{p}_y \frac{\hat{y}}{\hat{r}} + \frac{\hat{z}}{\hat{r}} \hat{p}_z + \hat{p}_z \frac{\hat{z}}{\hat{r}} \right)
\]

We can now plunge into rewriting the Hamiltonian. With much algebra, we can show that:

\[ \hat{H} = \frac{\hat{p}_r^2}{2m} + \frac{1}{2m \hat{r}^2} \frac{\hat{L}^2}{\hat{L}^2} + V(\hat{r}) \]

and we have rewritten the Hamiltonian in terms of a radial momentum operator and an angular momentum operator.