Orbital Angular Momentum

Recall that in spherical co-ordinates, the angular momentum operators are:

\[
\hat{L}_x \propto i\hbar \left[ \sin\phi \frac{\partial}{\partial \theta} + \cos\theta \cos\phi \frac{\partial}{\partial \phi} \right]
\]

\[
\hat{L}_y \propto i\hbar \left[ -\cos\phi \frac{\partial}{\partial \theta} + \cos\theta \sin\phi \frac{\partial}{\partial \phi} \right]
\]

\[
\hat{L}_z \propto -i\hbar \frac{\partial}{\partial \phi}
\]

We can use these to determine expectation values.

Example: Ignore for a moment, radial component of the wavefunction and consider

\[
\Psi(\theta, \phi) = \frac{1}{\sqrt{4\pi}} e^{im\phi}
\]

where m is constant. One can easily show that this is normalized.

a) Show that this is an eigenstate of \( \hat{L}_z \). Determine the eigenvalue.

b) Determine \( \langle L_z \rangle \) and \( (\Delta L_z)^2 \)

c) Determine \( \langle L_x \rangle \).

Answers

a) \( L_z |\psi\rangle \langle \psi| = \frac{m}{\hbar} \Psi(\theta, \phi) \quad \text{or} \quad m|\psi\rangle \)

This has eigenvalue \( m \).

b) Since it is an eigenstate, the expectation value is the same as the eigenvalue. Thus \( \langle L_z \rangle = m \hbar \) and \( \Delta L_z = 0 \).
c) $\langle L_x \rangle = \langle \Psi | \hat{L}_x | \Psi \rangle$

\[
= \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \; \Psi^* (\theta, \phi) \; \hat{L}_x \; \Psi (\theta, \phi)
\]

\[
= \frac{1}{4\pi} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \; e^{-im\phi} \; i\hbar \frac{\sin \theta}{\sin \theta} \cos \phi \; i\hbar \cos \phi \; e^{im\phi}
\]

\[
= -\frac{m\hbar}{4\pi} \int_0^\pi d\theta \cos \theta \int_0^{2\pi} d\phi \; \cos \phi
\]

\[
= 0
\]

\[
\Rightarrow \langle L_x \rangle = 0.
\]

Note that one requirement that this eigenstate has to satisfy is:

\[
e^{im\phi} \big|_{\phi=0} = e^{im\phi} \big|_{\phi=2\pi}
\]

\[
\Rightarrow 1 = e^{im2\pi}
\]

Thus $m$ must be an integer. This will emerge in a more general way from a complete treatment of angular momentum.

**Angular momentum eigenvalues.**

Recall that the angular momentum operators satisfy

\[
[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z
\]

\[
[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x
\]

\[
[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y
\]

\[
[\hat{L}_i, \hat{L}_i] = 0 \quad \text{where} \quad i = x, y, z
\]

and

\[
\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2
\]
The spin operators, $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$, satisfy similar relations and we are led to a generalized notion of angular momentum. Thus we consider three operators $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ such that

$$
\begin{align*}
\left[ \hat{J}_x, \hat{J}_y \right] &= i\hbar \hat{J}_z \\
\left[ \hat{J}_y, \hat{J}_z \right] &= i\hbar \hat{J}_x \\
\left[ \hat{J}_z, \hat{J}_x \right] &= i\hbar \hat{J}_y
\end{align*}
$$

and when we discuss orbital angular momentum we use $\hat{J} = \hat{L}$. For spin angular momentum $\hat{J} = \hat{S}$. Again we define

$$\hat{\mathbf{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

and it is easy to show

$$\left[ \hat{\mathbf{J}}^2, \hat{J}_i \right] = 0 \quad i = x, y, z.$$

Thus we attempt to simultaneously find eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$:

$$\hat{\mathbf{J}}^2 |j, m\rangle = \Lambda \langle j | j, m\rangle$$

$$\hat{J}_z |j, m\rangle = \lambda m |j, m\rangle$$

where $|j, m\rangle$ are normalized.

In what follows we will show that the allowed values of $m$ are:

$$j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

$$m = -j, -j+1, \ldots, j-1, j$$

such that

$$\hat{\mathbf{J}}^2 |j, m\rangle = j(j+1) \hbar^2 |j, m\rangle$$

$$\hat{J}_z |j, m\rangle = \lambda m |j, m\rangle.$$
Example: (Spin-1 particle) A spin-1 particle is one which yields one of three possible values when the \( z \)-component of its spin is measured.

<table>
<thead>
<tr>
<th>( S_z ) value</th>
<th>state ( \psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +\hbar )</td>
<td>( 1\rangle )</td>
</tr>
<tr>
<td>0</td>
<td>( 0\rangle )</td>
</tr>
<tr>
<td>( -\hbar )</td>
<td>( -1\rangle )</td>
</tr>
</tbody>
</table>

We can represent the states by:

\[
1\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad 0\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad -1\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

It follows that the operator for the \( z \)-component of spin is:

\[
\hat{S}_z = \hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

As an exercise you can verify that \( 1\rangle, 0\rangle, -1\rangle \) are eigenstates of the \( \hat{J}_z \) operator.

Additionally, it emerges that:

\[
\hat{J}_x = \frac{1}{\sqrt{2}} \hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_y = \frac{1}{\sqrt{2}} \hbar \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & i & 0 \end{pmatrix}
\]

Exercise: Show that \( \hat{J}_x, \hat{J}_y, \hat{J}_z \) satisfy the commutation relations for orbital angular momentum operators.

Exercise: Determine \( \hat{J}^2 \) for spin-1 and show that:

\[
\hat{J}^2 = 2\hbar^2 \hat{1}.
\]
The first step in determining angular momentum eigenvalues is to establish constraints on them.

**Constraining \( \Lambda_j \)**

It is natural to expect

\[
\Lambda_j > 0
\]

**Proof:**

\[
\langle j, m | \hat{J}^2 | j, m \rangle = \sum_{i=x,y,z} \langle j, m | \hat{J}_i^2 | j, m \rangle
\]

But \( \hat{J}_i \) is Hermitian and thus

\[
\hat{J}_i^2 = \hat{J}_i^+ \hat{J}_i
\]

\[
\Rightarrow \langle j, m | \hat{J}_i^+ \hat{J}_i | j, m \rangle \text{ is just an inner product of } \hat{J}_i | j, m \rangle \text{ with itself. This cannot be negative and thus establishes the result.}
\]

We can picture the angular momentum in terms of a vector with length \( \sqrt{\Lambda_j} \), but whose components are not yet pinned down. So its tip lies somewhere on the sphere of radius \( \sqrt{\Lambda_j} \). 

![Diagram of angular momentum vector on a sphere](image)
Constraining $\lambda_m$ for fixed $\Lambda_j$

Suppose that $\Lambda_j$ is fixed. What can be said about the $z$ component $\lambda_m$.

What about $\langle J_x \rangle, \langle J_y \rangle, (\Delta J_x)^2, (\Delta J_y)^2$ for these eigenstates?

We suspect that $\lambda_m ^2 \leq \Lambda_j$ and will prove this. This requires some knowledge of $\langle J_x \rangle, \langle J_y \rangle$

**Theorem:** For the state $|jm\rangle$, $\langle J_x \rangle = \langle J_y \rangle = 0$ and

$$\begin{align*}
(\Delta J_x)^2 &= \langle J_x^2 \rangle \\
(\Delta J_y)^2 &= \langle J_y^2 \rangle
\end{align*}$$

**Proof:**

$$\langle j,m| [\hat{J}_y, \hat{J}_z] |jm\rangle = i \hbar \frac{\langle j,m| J_x |jm\rangle}{\langle J_x \rangle}$$

$$\langle j,m| \hat{J}_x \hat{J}_y |jm\rangle - \langle j,m| \hat{J}_z \hat{J}_y |jm\rangle = i \hbar \langle J_x \rangle$$

$$\lambda_m \langle j,m| \hat{J}_x |jm\rangle - \Lambda_j \langle j,m| \hat{J}_x |jm\rangle = i \hbar \langle J_x \rangle$$

$$\Rightarrow \langle J_x \rangle = 0 \Rightarrow (\Delta J_x)^2 = \langle J_x^2 \rangle - \langle J_x \rangle^2 = \langle J_x^2 \rangle$$

Similarly for the $y$ component.

This gives the following geometric picture of the state $|jm\rangle$.

![Diagram](image)
Now
\[ \langle j m | \hat{J}^2 | j m \rangle = \langle j m | \hat{J}_x^2 | j m \rangle + \langle j m | \hat{J}_y^2 | j m \rangle + \langle j m | \hat{J}_z^2 | j m \rangle. \]

\[ \Rightarrow \lambda_j = (\Delta J_x)^2 + (\Delta J_y)^2 + \lambda_m. \]

**Exercise:** Prove this

Thus

\[ \lambda_m^2 = \lambda_j - (\Delta J_x)^2 - (\Delta J_y)^2. \]

To preserve \(\lambda_m\) to total angular motion squared

and unless \((\Delta J_x)^2 = (\Delta J_y)^2 = 0\) we will have

\[ -\sqrt{\lambda_j} < \lambda_j < \sqrt{\lambda_j}. \]

without equality. So we need to consider uncertainties for the state \(|j m\rangle\).

Can they both be zero? When is \((\Delta J_x)^2 + (\Delta J_y)^2\) smallest? What is smallest value. Here we need to return to the uncertainty principle:

\[ [\hat{J}_x, \hat{J}_y] = \frac{i \hbar}{2} \hat{J}_z. \]

\[ \Rightarrow \Delta \hat{J}_x \Delta \hat{J}_y = \frac{\hbar}{2} \left| \langle j m | [\hat{J}_x, \hat{J}_y] | j m \rangle \right| \]

\[ = \frac{\hbar}{2} \left| \lim_{j \to 0} \langle j m | \hat{J}_z | j m \rangle \right| \]

\[ \Rightarrow \Delta \hat{J}_x \Delta \hat{J}_y = \frac{\hbar}{2} |\lambda_m|. \]

and unless \(|\lambda_m| = 0\) we get that both \(\Delta \hat{J}_x, \Delta \hat{J}_y \neq 0\).
This is partly expected, since if $\Delta J_x = 0$ both the $x$ and $z$ components would be known with certainty and one knows that this is not possible. Similarly for $\Delta J_y$. The necessity that both of these are not-zero means that we can never have the $z$ component value exactly equal to the magnitude. How different can these be? Here consider

$$
\lambda_m^2 = \Lambda_j - \left( (\Delta J_x)^2 + (\Delta J_y)^2 \right)
$$

and we need to answer: "What is the minimum value of $(\Delta J_x)^2 + (\Delta J_y)^2$ given that $\Delta J_x \Delta J_y > \frac{h^2}{2} |\lambda_m|$?"

Suppose that $\lambda_m$ is fixed. Then consider the plot of $\Delta J_x$ and $\Delta J_y$.

$\Delta J_x \Delta J_y = \frac{h^2}{2} |\lambda_m|$

$\Delta J_x \Delta J_y = \frac{h^2}{2} |\lambda_m|$ corresponds to a hyperbola, and the values of $\Delta J_x$ and $\Delta J_y$ must correspond to points beyond the hyperbola. On the other hand $(\Delta J_x)^2 + (\Delta J_y)^2$ corresponds to the distance from the origin. The closest point on the origin to the hyperbola lies along the 45° line. Here $\Delta J_x = \Delta J_y$. Thus for this point

$$(\Delta J_x)(\Delta J_y) > \frac{h^2}{2} |\lambda_m|$$

$$(\Delta J_x)^2 > \frac{h^2}{2} |\lambda_m|$$

Similarly $(\Delta J_y)^2 > \frac{h^2}{2} |\lambda_m|$. 
These imply that
\[(\Delta x)^2 + (\Delta y)^2 \geq \frac{\hbar}{\lambda_m} \lambda_m.\]

and
\[\lambda_m^2 \leq \lambda_j - \frac{\hbar}{\lambda_m}.\]

d\[\lambda_m^2 + \frac{\hbar}{\lambda_m} \leq \lambda_j\]

d\[\lambda_m^2 + \frac{\hbar}{\lambda_m} + \frac{\hbar^2}{4} \leq \lambda_j + \frac{\hbar^2}{4}.\]

d\[(\lambda_m + \frac{\hbar}{2})^2 \leq \lambda_j + \frac{\hbar^2}{4}.\]

Now suppose \(\lambda_m > 0\). The above gives:

\[-\sqrt{\lambda_j + \frac{\hbar^2}{4}} \leq (\lambda_m + \frac{\hbar}{2}) \leq \sqrt{\lambda_j + \frac{\hbar^2}{4}} \Rightarrow -\frac{\hbar}{2} - \sqrt{\lambda_j + \frac{\hbar^2}{4}} \leq \lambda_m \leq \sqrt{\lambda_j + \frac{\hbar^2}{4}} - \frac{\hbar}{2}\]

But for \(\lambda_m < 0\) we get

\[-\sqrt{\lambda_j + \frac{\hbar^2}{4}} \leq (-\lambda_m + \frac{\hbar}{2}) \leq \sqrt{\lambda_j + \frac{\hbar^2}{4}} \Rightarrow \frac{\hbar}{2} - \sqrt{\lambda_j + \frac{\hbar^2}{4}} \leq \lambda_m \leq 0\]

Thus we get:

\[
\frac{\hbar}{2} - \sqrt{\lambda_j + \frac{\hbar^2}{4}} \leq \lambda_m \leq \sqrt{\lambda_j + \frac{\hbar^2}{4}} - \frac{\hbar}{2}
\]

quantum.

which bounds the \(z\)-component. Note that you can show that (exercise)

\[\sqrt{\lambda_j + \frac{\hbar^2}{4}} - \frac{\hbar}{2} \leq \sqrt{\lambda_j} \]

\[\frac{\hbar}{2} - \sqrt{\lambda_j + \frac{\hbar^2}{4}} \geq -\sqrt{\lambda_j}\]
Thus the angular momentum vector cannot attain the same values that it would classically, since classically one would have \(-\sqrt{\hbar j} \leq \lambda_m \leq \sqrt{\hbar j}\).

These can be illustrated as follows for the cases where \(\lambda_m\) is closest to \(\sqrt{\hbar j}\) or \(-\sqrt{\hbar j}\).

\[
\lambda_m \in \text{ here}
\]

\[
\sqrt{\hbar j}
\]

\[
\lambda_j
\]

\[
\text{ sphere of radius } \sqrt{\hbar j}
\]

\[
\text{ maximum } z\text{-component}
\]

Example: Consider a classical particle. We frequently find that the orbital angular momentum is \(\vec{L} = L_0 \hat{\mathbf{z}}\). Then \(|\vec{L}^2| = L_0^2\) and \(L_2 = L_0\).

Now \(|\vec{L}^2| \rightarrow \lambda_j\)

\[L_2 \rightarrow \lambda_m\]

and this would require \(\lambda_j = \lambda_m^2\). But we know \(\sqrt{\hbar j} \leq \lambda_m\). How can we reconcile this?

The answer is that for \(\lambda_j \gg \frac{\hbar}{4}\), we have

\[
\sqrt{\lambda_j + \frac{\hbar^2}{4}} - \frac{\hbar}{2} \approx \sqrt{\lambda_j}
\]

and this regains the classical result.
We now note that it will be convenient to rewrite $\Lambda_j$ as follows:

$$\Lambda_j = A_j (A_j + 1) h^2$$

for some $A_j > 0$.

Exercise: Show that $A_j = \frac{-1 + \sqrt{1 + 4 A_j / h^2}}{2}$ is the one solution which guarantees that $\Lambda_j > 0$. Show that $A_j > 0$.

The bound then becomes:

$$-A_j h \leq \lambda_m \leq A_j h$$

Exercise: Show this.