B. Simple combinations of S-G experiments.

- Suppose that the magnetic field is oriented in the $\hat{n}$ direction. Then we diagram the basic experiment by:

\[
\begin{array}{c}
\text{Box containing magnets} \\
\text{with gradient in $\hat{n}$ direction.}
\end{array}
\]

\[
\begin{array}{c}
in \\
\rightarrow S_n = \frac{1}{2} \\
\downarrow \\
on \\
\rightarrow S_n = -\frac{1}{2}
\end{array}
\]

- We note that regardless of the direction of choice, $\hat{n}$, the will always be one of just two possible outcomes for an S-G experiment.
- Which we get can depend on the particle's prior history.
- In the following, we discuss sequences of S-G experiments and their outcomes. These are all thought experiments but analogous versions can be done using photons and their polarization states. More sophisticated features of QM have also been tested by neutron interferometry or trapped-ions.

Experiment 1. Repeated SG measurements of the same time

Suppose that we perform successive S-G experiments in the same direction.
- Consider a single particle which first passes through the left SG apparatus and then through one of those on the right.

- If you get $S_z = +\frac{1}{2}$ after the first, then you will only get $S_z = +\frac{1}{2}$ on the second and never $S_z = -\frac{1}{2}$. This is independent of the particle's state prior to the first SG apparatus. Similarly for $S_z = -\frac{1}{2}$. The measurement is repeatable.

- So, as far as all future measurements of the particle's $z$-component are concerned, if you only measure $z$-component + do nothing else, the particle has a definite value of $S_z$ after the first SG apparatus. We denote the state of this particle by a “ket”:

$$|\uparrow\rangle \leftrightarrow \text{when you measure } S_z \text{ you always get } S_z = +\frac{1}{2} \text{ with certainty (you will never obtain } S_z = -\frac{1}{2})$$

$$|\downarrow\rangle \leftrightarrow \text{never obtain } S_z = +\frac{1}{2} \text{- always } S_z = -\frac{1}{2}.$$ 

and clearly these are different obtained. Note that the contents of the ket are a label. So in general we have:

$$|\uparrow\rangle \leftrightarrow \text{when you measure } S_n \text{ (i.e. place the particle in SG$n$)}$$

$$\text{you get } S_n = +\frac{1}{2} \text{ with certainty.}$$

$$|\downarrow\rangle \leftrightarrow \text{ never obtain } S_n = +\frac{1}{2} \text{ with certainty.}$$

- For every unit vector pair $|\uparrow\rangle, |\downarrow\rangle$, there are two states $|\uparrow\rangle, |\downarrow\rangle$. There are infinitely many states available to this system.

- Also note the in some cases in quantum mechanics, measurement outcomes can be predicted with certainty. So a state

$$|\uparrow\rangle \rightarrow \text{ SG$x$ } \rightarrow S_x = +\frac{1}{2} \text{ with certainty.}$$
Experiment 2  SG measurements of different types

Now suppose that one combines SG measurements where the apparatus is oriented in different directions. For example:

\[
\begin{array}{c}
\text{mixture} \quad \text{SG} \frac{z}{x} \\
\begin{array}{c}
S_z = +\frac{\pi}{6} \\
1+2\rangle \\
S_z = -\frac{\pi}{6} \\
1-2\rangle
\end{array} \\
\begin{array}{c}
\text{SGx} \\
S_x = +\frac{\pi}{6} \\
1+x\rangle \\
S_x = -\frac{\pi}{6} \\
1-x\rangle
\end{array}
\end{array}
\]

What do we find experimentally. Consider particles emerging from the first SG\(z\) apparatus with \(S_z = +\frac{\pi}{6}\) (i.e., in the state \(|1+2\rangle\)). Even though these all give the same outcome for the SG\(z\) measurements, they do not all give the same outcome for SG\(x\). In fact, it seems that the outcomes are randomly distributed. Many repeated experiments will verify this, and we can determine probabilities with which outcomes of the second measurement occur. So

<table>
<thead>
<tr>
<th>outcome (S_x)</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_x = +\frac{\pi}{6})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(S_x = -\frac{\pi}{6})</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>

Similarly, \(1-2\rangle\) into SG\(x\)

Furthermore, once you actually choose the \(|1+2\rangle\) beam, the details of what happened prior to the SG\(z\) are irrelevant for these statistics.

We now consider a general case where an SG\(z\) measurement is followed by a SG\(x\) measurement.
After the $SG\hat{n}$ measurement, the particles are in the state $|+\hat{n}\rangle$ or $|1-\hat{n}\rangle$.

Then consider taking a particle in the $|+\hat{n}\rangle$ beam and doing a $SG\hat{n}$ measurement

$$|+\hat{n}\rangle \text{ followed by } SG\hat{n} \rightarrow$$

<table>
<thead>
<tr>
<th>outcome</th>
<th>prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n = +t/2$</td>
<td>$\frac{1}{2} (</td>
</tr>
<tr>
<td>$S_n = -t/2$</td>
<td>$\frac{1}{2} (</td>
</tr>
</tbody>
</table>

The case of $|1-\hat{n}\rangle$ is accounted for by replacing $\hat{n}$ by $-\hat{n}$ in these formulae.

**Consistency check:**

1) $|+\hat{n}\rangle$ followed by $SG\hat{n}$

- $S_m = +t/2$ certainty.

2) $|1-\hat{n}\rangle$ followed by $SG\hat{n}$

- $S_m = -t/2$ certainty.

Now you could object and say that one needs a ket label that reflects what happens for successive measurements. For example, if we do $SG\hat{x}$ followed by $SG\hat{y}$ we can get the following

$$SG\hat{x} \rightarrow \begin{array}{c} S_n = +t/2 \\ SG\hat{y} \end{array}$$

Maybe we should use $|+\hat{x},+\hat{y}\rangle$ to reflect what will happen in both measurements. Let's consider placing another $SG\hat{y}$ measurement after the $SG\hat{y}$.

Q. What are the possible outcomes and probabilities?

A. We know that after the $SG\hat{y}$ we have $|+\hat{y}\rangle$ beam. The earlier history is irrelevant. So we can calculate using the formula above:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_y = +t/2$</td>
<td>1</td>
</tr>
<tr>
<td>$S_y = -t/2$</td>
<td>0</td>
</tr>
</tbody>
</table>
Now repeat this with $SG\hat{x}$ after $SG\hat{y}$.

<table>
<thead>
<tr>
<th>outcome</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Sx = +\frac{\hbar}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$Sx = -\frac{\hbar}{2}$</td>
<td>$\frac{1}{2}$.</td>
</tr>
</tbody>
</table>

So we have

So whereas you could be certain about measurement outcomes after the first $SG\hat{x}$, inserting $SG\hat{y}$, means you can no longer be confident about outcomes of future $SG\hat{x}$. Thus the state changes from

\[ |+x\rangle \rightarrow |+y\rangle \]  

in upper beam $SG\hat{y}$

and loses information about its history prior to $SG\hat{y}$. 