Einstein–Podolsky–Rosen paradox

Consider the state of two spin-$\frac{1}{2}$ quantum systems, labelled A and B:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |+z\rangle_A |-z\rangle_B - |-z\rangle_A |+z\rangle_B \right)$$

Suppose that the $z$-component of spin of each particle is measured. The outcomes and probabilities are:

<table>
<thead>
<tr>
<th>$S_z^A$</th>
<th>$S_z^B$</th>
<th>Prob</th>
<th>state after</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$</td>
</tr>
<tr>
<td>$-\frac{1}{2}$</td>
<td>$+\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$</td>
</tr>
</tbody>
</table>

In fact, only one party actually has to do the measurement in order to be sure of both outcomes. This is evident from the table above: $S_z^A$ and $S_z^B$ are always opposite (they are perfectly correlated).

The paradox is constructed by imagining that the particles are separated by a substantial distance.

EPR then assumed that a local action on A, such as a measurement of any component of A's spin, cannot have any effect on B.
But note that if one measures just the $z$-component of $A$, then we immediately know the $z$-component of $B$. Then EPR claimed, this can only mean that $B$ had that value all along. For example, suppose that $S_z^A = +\frac{\hbar}{2}$. According to EPR this would mean that $B$ definitely has $S_z^B = -\frac{\hbar}{2}$ before any measurement is done. In the language of kets one would say that the state of $B$ is $1-\frac{\hbar}{2}$. The first potential mystery is that it is not clear what $A$ will have to obtain to begin with and that $B$ will somehow have to adjust remotely and instantaneously. But one can get around this by saying that the state $1\{+\}$ means that really $A$ and $B$ are in one of the following two states before anything is done

$$1+\frac{\hbar}{2}A\frac{1}{2}B \quad \text{or} \quad 1-\frac{\hbar}{2}A\frac{1}{2}B$$

Then there is no need for any mysterious long range signal. However, we can try to work with other bases. Recall that

$$1+\frac{\hbar}{2} = \frac{1}{\sqrt{2}} \left\{ 1+\frac{\hbar}{2} + 1-\frac{\hbar}{2} \right\}$$

$$1-\frac{\hbar}{2} = \frac{1}{\sqrt{2}} \left\{ 1+\frac{\hbar}{2} - 1-\frac{\hbar}{2} \right\}$$

and also

$$1+\frac{\hbar}{2} = \frac{1}{\sqrt{2}} \left\{ e^{i\pi/4} 1+\frac{\hbar}{2} + e^{-i\pi/4} 1-\frac{\hbar}{2} \right\}$$

$$1-\frac{\hbar}{2} = \frac{1}{\sqrt{2}} \left\{ e^{i\pi/4} 1+\frac{\hbar}{2} - e^{-i\pi/4} 1-\frac{\hbar}{2} \right\}$$
Exercise: Show that

\[ |ψ⟩ = \frac{1}{\sqrt{2}} \left\{ 1+χA 1-x⟩_B - 1-x⟩_A 1+χ⟩_B \right\} \]

\[ |ψ⟩ = \frac{1}{\sqrt{2}} \left\{ 1+y⟩A 1-y⟩_B - 1-y⟩A 1+y⟩_B \right\}. \]

In fact, it is not hard to show that for any unit vector \( \hat{n} \):

\[ |ψ⟩ = \frac{1}{\sqrt{2}} \left\{ 1+\hat{n}⟩A 1-\hat{n}⟩_B - 1-\hat{n}⟩A 1+\hat{n}⟩_B \right\}. \]

Now this means that if one measures any component of spin for \( A \) and gets:

\[ S^A_n = +\frac{1}{2} \]

then one has to obtain \( S^B_n = -\frac{1}{2} \).

The key idea in the EPR argument is to combine statements for different components of spin measurements. Starting with measuring a \( z \)-component for \( A \), whatever result we get this fixes a state of \( B \). But since local actions on \( A \) cannot affect \( B \), we would say that \( B \) had this state prior to the measurement. So \( B \) had a definite value of \( S_z \) all along. But we could have chosen to measure the \( x \) component and would use the same argument to conclude that \( B \) had a definite value of \( S_x \). But we have not done anything to \( B \) at all and the only reasonable conclusion is that \( B \) had definite values for \( S_x, S_z \) (and any other component as well). But we know that the uncertainty principle prohibits this since, in general:

\[ ΔS_x ΔS_z ≠ 0. \]

Thus one reaches a contradiction.
Hidden variable theories.

It seems clear that what Einstein envisaged was that quantum mechanics is an incomplete theory. For example, how it can only tell us about one of possibly many variables needed to describe the physical situation, $S_x$, $S_y$ or $S_z$.

But the above paradox would indicate that this is perhaps not true and that somehow each particle has a record of values of $S_x, S_z$, etc... that will emerge when these components are measured. Suppose that $S_z^A = +\frac{1}{2}$. Then it is clear that $S_z^B = -\frac{1}{2}$. But what about $S_x^B$? We could have either $S_x^B = +\frac{1}{2}$ or $S_x^B = -\frac{1}{2}$. Einstein would prefer to list these attributes:

$$\{ S_z^B = +\frac{1}{2}, S_x^B = +\frac{1}{2} \} \sim \{ +\frac{1}{2}, +\frac{1}{2} \}_B$$

Or $$\{ S_z^B = +\frac{1}{2}, S_x^B = -\frac{1}{2} \} \sim \{ +\frac{1}{2}, -\frac{1}{2} \}_B$$

and by the EPR paradox, the B particle is in one of these arrangements. It's just that quantum mechanics does not tell us which one. So the variables are "hidden" from quantum mechanics. This would be an example of a hidden variables theory. The question now is whether we can tell if a hidden variables theory is reasonable or not.

Measuring correlations between outcomes.

It turns out that by considering correlations between outcomes of measurements, we can determine if a hidden variables theory is reasonable. One simple possibility involves measuring the component of A's spin along the $\hat{a}$ direction and measuring B's spin along the $\hat{b}$ direction.
We always obtain

\[ S_a^A = \pm \frac{1}{2} \text{ or } -\frac{1}{2} \]

\[ S_b^B = \pm \frac{1}{2} \text{ or } -\frac{1}{2} \]

and we ignore the factor of \( \frac{1}{2} \) and focus on the signs. What we are interested in is the probability that \( S_a^A \) gives + and \( S_b^B \) gives +.

Call this:

\[ P(+\hat{a}, +\hat{b}) = \text{prob}(S_a^A = +\frac{1}{2} \text{ and } S_b^B = +\frac{1}{2}) \]

Similarly, we can construct \( P(+\hat{a}, -\hat{b}) \), etc.

**Examples**: Suppose that \( \hat{a} = \hat{b} = \hat{\varepsilon} \). Then quantum mechanics predicts that

\[ P(+\varepsilon, +\varepsilon) = 0 \]

\[ P(+\varepsilon, -\varepsilon) = \frac{1}{2} \]

\[ P(-\varepsilon, +\varepsilon) = \frac{1}{2} \]

\[ P(-\varepsilon, -\varepsilon) = 0 \]

**Exercise**: Show that

\[ P(+\hat{x}, +\hat{x}) = P(-\hat{x}, -\hat{x}) = 0 \]

\[ P(+\hat{x}, -\hat{x}) = P(-\hat{x}, +\hat{x}) = 0 \]

**Example**: Suppose that \( \hat{a} = +\hat{\varepsilon} \), \( \hat{b} = +\hat{x} \). Then to assess this quantum mechanically,

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |\hat{\varepsilon}\rangle_A |\hat{x}\rangle_B - |\hat{x}\rangle_A |\hat{\varepsilon}\rangle_B \right) \]

\[ = \frac{1}{\sqrt{2}} \left\{ (|\hat{\varepsilon}\rangle_A |\hat{x}\rangle_B - |\hat{x}\rangle_A |\hat{\varepsilon}\rangle_B) - (|\hat{x}\rangle_A |\hat{\varepsilon}\rangle_B + |\hat{\varepsilon}\rangle_A |\hat{x}\rangle_B) \right\} \]

\[ = \frac{1}{2} \left\{ (|\hat{\varepsilon}\rangle_A |\hat{\varepsilon}\rangle_B - |\hat{\varepsilon}\rangle_A |\hat{\varepsilon}\rangle_B) - (|\hat{x}\rangle_A |\hat{x}\rangle_B + |\hat{x}\rangle_A |\hat{x}\rangle_B) \right\} \]
Thus quantum mechanics predicts that

\[ P(+\hat{a}, +\hat{b}) = \frac{1}{4} \]
\[ P(+\hat{a}, -\hat{b}) = \frac{1}{4} \]
\[ P(-\hat{a}, +\hat{b}) = \frac{1}{4} \]
\[ P(-\hat{a}, -\hat{b}) = \frac{1}{4}. \]

Generally we can determine these probabilities quantum mechanically via.

\[ P(+\hat{a}, +\hat{b}) = |\langle +\hat{a}| +\hat{b}| \psi \rangle|^2 \]

But

\[ |\psi\rangle = \frac{1}{\sqrt{2}} \left( |+\hat{a}\rangle |1-\hat{a}\rangle - |-\hat{a}\rangle |1+\hat{a}\rangle \right). \]

\[ \Rightarrow P(+\hat{a}, +\hat{b}) = \frac{1}{2} |\langle +\hat{b}|-\hat{a}\rangle|^2 \]

But recall that for any unit vectors \( \hat{a}, \hat{m} \):

\[ |\langle +\hat{a}| +\hat{m}\rangle|^2 = \frac{1}{2} (1 + \cos \theta) = \cos^2 \theta / 2 \]

where \( \theta = \hat{a}, \hat{m} \) have let \( \theta_{ab} \) be the angle between \( +\hat{a} \) and \( +\hat{b} \)
So the angle between $+\hat{b}$ and $-\hat{a}$ is $\frac{\pi}{2} + \theta_{ab}$, and so:

$$P(+\hat{a}, +\hat{b}) = \frac{1}{2} \cos^2 \left( \frac{\pi + \theta_{ab}}{2} \right)$$

$$= \frac{1}{2} \sin^2 \frac{\theta_{ab}}{2}.$$ 

Similar arguments lead to: quantum mechanics predicts that:

$$P(+\hat{a}, +\hat{b}) = \frac{1}{2} \sin^2 \left( \frac{\theta_{ab}}{2} \right)$$

$$P(+\hat{a}, -\hat{b}) = \frac{1}{2} \cos^2 \left( \frac{\theta_{ab}}{2} \right)$$

$$P(-\hat{a}, +\hat{b}) = \frac{1}{2} \cos^2 \left( \frac{\theta_{ab}}{2} \right)$$

$$P(-\hat{a}, -\hat{b}) = \frac{1}{2} \sin^2 \left( \frac{\theta_{ab}}{2} \right).$$

**Exercise:** Demonstrate these.

Now consider the product $S_a \hat{A} S_b \hat{B}$. We can easily show that quantum mechanics predicts that the expectation value is:

$$E_{\text{quantum}} = \langle S_a \hat{A} S_b \hat{B} \rangle = \frac{\hbar^2}{4} \left( \sin^2 \frac{\theta_{ab}}{2} - \cos^2 \frac{\theta_{ab}}{2} \right)$$

$$E_{\text{quantum}} = -\frac{\hbar^2}{4} \cos \theta_{ab}.$$
What would a hidden variables theory predict? This depends on the nature of the hidden variables. A simple example would have one associate a vector \( \vec{s} \) with \( A \) - represents \( A \)'s spin.

Necessarily the spin vector for \( B \) is \( -\vec{s} \). Now assume that all orientations of \( \vec{s} \) are equally likely and suppose that we measure the projection of \( \vec{s} \) along \( \hat{a} \) for particle \( A \) and \( -\vec{s} \) along \( \hat{b} \) for particle \( B \). We are just interested in the sign of \( \theta \), these and also need to rescale so that they are compatible with the measurement scale \( \frac{\pi}{2} \).

Let \( \phi \) be the angle between \( \vec{s} \) and \( \hat{a} \).

Then we get the following outcomes depending on \( \theta \):

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>Product of signs of components and ( \frac{\pi}{4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\frac{\pi}{2} + \theta ) to ( \frac{\pi}{2} )</td>
<td>(-e^{i\frac{\pi}{4}})</td>
</tr>
<tr>
<td>( \frac{\pi}{2} ) to ( \frac{\pi}{2} + \theta )</td>
<td>( +e^{i\frac{\pi}{4}})</td>
</tr>
<tr>
<td>( \theta + \frac{3\pi}{2} ) to ( 3\pi/2 )</td>
<td>(-e^{i\frac{\pi}{4}})</td>
</tr>
<tr>
<td>(-\frac{\pi}{2} ) to (-\frac{\pi}{2} + \theta )</td>
<td>( +e^{i\frac{\pi}{4}})</td>
</tr>
</tbody>
</table>
But we have to average this over all possible angles $\phi$. This gives the hidden variables expectation value:

$$E_{hv} = \frac{1}{2\pi} \left[ \int_{-\frac{\pi}{2} + \Theta_{ab}}^{\frac{\pi}{2}} d\phi + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\phi - \int_{\frac{\pi}{2} + \Theta_{ab}}^{\pi} d\phi - \int_{-\frac{\pi}{2}}^{-\Theta_{ab}} d\phi \right] \frac{C}{4}$$

$$= \frac{\hbar^2}{4} \frac{1}{2\pi} \left[ - (\pi - \Theta_{ab}) + \Theta_{ab} - (\pi + \Theta_{ab}) + (\frac{\pi}{2} + \Theta_{ab}) + \pi \right]$$

$$= \frac{\hbar^2}{4} \frac{1}{2\pi} \left[ - 2\pi + 4\Theta_{ab} \right]$$

$$E_{hv} = \frac{\hbar^2}{4} \left[ \frac{2\Theta_{ab}}{\pi} - 1 \right]$$

But note that for various values of $\Theta_{ab}$, this gives different results than quantum theory predicts.
Bell's inequalities

One may try to modify the previous attempt at a hidden variables theory, so that it always agrees with quantum mechanics. In 1964, John Bell demonstrated that, if one assumes locality and carries out pairs of measurements oriented along axes \( \hat{a}, \hat{b}, \) and \( \hat{c} \) then the expectation values from the various pairs for a hidden variables theory gives:

\[
|E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c})| \leq 1 + E_{HV}(\hat{b},\hat{c})
\]

"Bell inequality"

where now the factors of \( \hbar^2/4 \) are omitted from the expectation values. One can compare this to the equivalent statement for a quantum theory. Surprisingly one finds that for certain choices of \( \hat{a}, \hat{b}, \hat{c} \) the quantum expectation values do not satisfy the Bell inequality.

This has been tested using entangled photons and the Bell inequalities have always been violated!