Class Exercise 9: Quantum States
29 April 2005

1 Quantum states for spin-1/2 particles

a) For each of the following states $|\psi\rangle$ determine the associated bra $\langle \psi |$.

i) $|\psi\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{1}{\sqrt{2}} |-\hat{z}\rangle$

ii) $|\psi\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle - i \frac{1}{\sqrt{2}} |-\hat{z}\rangle$

b) For each of the following, calculate $\langle \phi | \psi \rangle$.

i) $|\psi\rangle = |+\hat{z}\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{1}{\sqrt{2}} |-\hat{z}\rangle$

ii) $|\psi\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle - i \frac{1}{\sqrt{2}} |-\hat{z}\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{1}{\sqrt{2}} |-\hat{z}\rangle$

Recall that

\[
\langle +\hat{z} | +\hat{z} \rangle = 1 \\
\langle +\hat{z} | -\hat{z} \rangle = 0 \\
\langle -\hat{z} | +\hat{z} \rangle = 0 \\
\langle -\hat{z} | -\hat{z} \rangle = 1.
\]

2 Quantum states and measurements

Suppose that a spin-1/2 particle is in the state $|\psi\rangle$ and that it is subjected to a Stern-Gerlach apparatus whose magnetic field is oriented in the direction of the unit vector $\hat{n}$. This effectively measures the component of the spin along $\hat{n}$. The various measurement outcomes occur as follows:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
<th>State after measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+\frac{\hbar}{2}$</td>
<td>$</td>
<td>\langle \hat{n}</td>
</tr>
<tr>
<td>$-\frac{\hbar}{2}$</td>
<td>$</td>
<td>\langle -\hat{n}</td>
</tr>
</tbody>
</table>

where $|+\hat{n}\rangle$ is the state such that the outcome is $+\frac{\hbar}{2}$ with certainty and $|-\hat{n}\rangle$ is the state such that the outcome is $-\frac{\hbar}{2}$ with certainty.

To actually apply this rule it is useful to adopt a particular basis with which to express the states; the most common choice is $\{ |+\hat{z}\rangle, |-\hat{z}\rangle \}$. In terms of this basis,

\[
|+\hat{a}\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{1}{\sqrt{2}} |-\hat{z}\rangle \\
|\hat{a}\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle - \frac{1}{\sqrt{2}} |-\hat{z}\rangle \\
|+\hat{y}\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{i}{\sqrt{2}} |-\hat{z}\rangle \\
|\hat{y}\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle - \frac{i}{\sqrt{2}} |-\hat{z}\rangle
\]
a) Suppose that a spin-1/2 particle initially in the state $|+z\rangle$. The $z$-component of the spin is measured. List the possible outcomes, the probabilities with which they occur and the state of the particle after the measurement. Repeat this for a particle initially in the state $|-z\rangle$.

b) Suppose that a particle is initially in the state $|+z\rangle$. The $x$-component of the spin is measured. List the possible outcomes, the probabilities with which they occur and the state of the particle after the measurement. Repeat this for a particle initially in the state $|-z\rangle$.

3 Sequences of measurements
Consider two possible measurement scenarios. In the first a particle prepared in one of the states $\{|+z\rangle, |-z\rangle\}$ and is subjected to a Stern-Gerlach apparatus oriented along $\hat{z}$.

In the second the particle is prepared in one of the same states and the measurement along $\hat{z}$ is preceded by a measurement along $\hat{x}$.

a) In each case assume that the initial state of the particle is $|+z\rangle$. Determine the possible measurement outcomes and the probabilities with which they occur in the first scenario. List all possible measurement outcomes and the probabilities with which they occur in the second scenario (there should be four cases).

b) Determine the probability with which the second scenario delivers the same outcome as the first.

c) Repeat the previous question for the initial state $|-z\rangle$. 

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