**Solutions to Complex Arithmetic Practice Problems**

Rectangular to polar problem set:

To convert to polar form, must find magnitude and phase.

1. mag. $= \sqrt{(18)^2 + (15)^2} = 23.4$  phase $= \tan^{-1}\left(\frac{15}{18}\right) = 0.695$ rad

   $\rightarrow 18 + j15 = 23.4e^{j0.695} = 23.4\angle 39.8°$

2. mag. $= \sqrt{(-3)^2 + (-21)^2} = 21.2$  phase $= \tan^{-1}\left(\frac{-21}{-3}\right) = 1.43 + \pi = 4.57$ rad

   (Note that some calculators don't place the result of the inverse tangent function in the correct quadrant when the real part is negative.)

   $\rightarrow -3 - j21 = 21.2e^{j4.57} = 21.2\angle -98°$

3. mag. $= \sqrt{(-3)^2 + (21)^2} = 21.2$  phase $= \tan^{-1}\left(\frac{21}{-3}\right) = -1.43 + \pi = 1.71$ rad

   $\rightarrow -3 + j21 = 21.2e^{j1.71} = 21.2\angle 98°$

4. mag. $= \sqrt{a^2 + b^2}$  phase $= \tan^{-1}\left(\frac{b}{a}\right)$

   $\rightarrow a + jb = \sqrt{a^2 + b^2}\, e^{\tan^{-1}(b/a)} = \sqrt{a^2 + b^2}\angle \tan^{-1}(b/a)$

5. mag. $= \sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1$  phase $= \tan^{-1}\left(\frac{\sin \omega t}{\cos \omega t}\right) = \tan^{-1}(\tan \omega t) = \omega t$

   $\rightarrow \cos \omega t + j \sin \omega t = e^{j\omega t}$ (Euler's formula!)

6. mag. $= \sqrt{\sin^2 \omega t + \cos^2 \omega t} = 1$

   phase $= \tan^{-1}\left(\frac{\cos \omega t}{\sin \omega t}\right) = \tan^{-1}\left[\frac{\cos(-\omega t)}{\sin(\omega t)}\right] = \tan^{-1}\left[\frac{\sin(-\omega t + \pi/2)}{\cos(\omega t - \pi/2)}\right] = \tan^{-1}\left[\frac{\sin(\pi/2 - \omega t)}{\cos(\pi/2 - \omega t)}\right]$

   $= \tan^{-1}[\tan(\pi/2 - \omega t)] = \pi/2 - \omega t$

   $\rightarrow \sin \omega t + j \cos \omega t = e^{j(\pi/2 - \omega t)} = 1\angle\left(90° - \omega t \frac{360°}{2\pi}\right)$ (this is correct phasor form)

   this is also equal to $e^{j(\pi/2 - \omega t)} = e^{j\pi/2} e^{-j\omega t} = je^{-j\omega t}$

7. mag. $= \sqrt{9^2\cos^2\omega t + 3^2\sin^2\omega t} = \sqrt{81\left(1-\sin^2\omega t\right) + 9\sin^2\omega t} = \sqrt{81 - 72\sin^2\omega t}$

(magnitude varies between 3 and 9, depending on the time $t$)

phase $= \tan^{-1}\left(\frac{3\sin\omega t}{9\cos\omega t}\right) = \tan^{-1}\left(\frac{1}{3}\tan\omega t\right) \neq \frac{1}{3}\omega t$

(phase angle does not vary linearly with time as in the case of $e^{j\omega t}$)

Polar to rectangular problem set:

1. $8e^{-j0.12} = 8\left[\cos(-0.12) + j\sin(-0.12)\right] = 8\left[\cos(0.12) - j\sin(0.12)\right] = 7.94 - j0.096$

2. $14\angle 132^\circ = 14\left[\cos(132^\circ) + j\sin(132^\circ)\right] = -9.37 + j10.4$

3. $-6\angle -85^\circ = -6\left[\cos(-85^\circ) + j\sin(-85^\circ)\right] = -6\left[\cos(85^\circ) - j\sin(85^\circ)\right] - 0.523 + j5.98$

4. $Ae^{j\omega t} = A\left(\cos\omega t + j\sin\omega t\right) = A\cos\omega t + jA\sin\omega t$

5. $Be^{-j(\omega t+\pi/6)} = B\left[\cos(-\omega t - \pi/6) + j\sin(-\omega t - \pi/6)\right]$

   $= B\left[\cos(\omega t + \pi/6) - j\sin(\omega t + \pi/6)\right] = B\cos\left(\omega t + \pi/6\right) - jB\sin\left(\omega t + \pi/6\right)$

6. $15e^{j(\omega t - \pi/2)} = 15e^{-j\pi/2}e^{j\omega t} = -j15e^{j\omega t} = -j15(\cos\omega t + j\sin\omega t)$

   $= -j15\cos\omega t - j^2 15\sin\omega t = 15\sin\omega t - j15\cos\omega t$

7. $e^{-j3\pi/2} = \cos\left(\frac{-3\pi}{2}\right) + j\sin\left(\frac{-3\pi}{2}\right) = 0 + j = j$

Express in proper polar form:

Proper polar form requires a complex number to be expressed in terms of a positive magnitude and a phase angle (phase can be expressed either in exponential form or in angular form; that is, either "mag $e^{j\,\text{phase}}$" or "mag$\angle$phase").

1. $-10e^{-j3\pi/2} = 10(-1)e^{-j3\pi/2} = 10e^{j\pi}e^{-j3\pi/2} = 10e^{j(\pi-3\pi/2)} = 10e^{-j\pi/2}$
   or $10e^{-j\pi}e^{-j3\pi/2} = 10e^{j(-\pi-3\pi/2)} = 10e^{-j5\pi/2} = 10e^{-j2\pi}e^{-j\pi/2} = 10e^{-j\pi/2}$

   The usual practice is to give phase values in the range $-\pi$ to $\pi$.

   Note that $e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1 + j0 = -1$ and that

$e^{-j\pi} = \cos(-\pi) + j\sin(-\pi) = \cos(\pi) - j\sin(\pi) = -1$ ($\pi$ rad $= 180°$).

Also note that $e^{j2\pi} = e^{-j2\pi} = 1$.

2. $Be^{-j\omega t - \alpha t} = Be^{-j\omega t}e^{-\alpha t} = Be^{-\alpha t}e^{-j\omega t}$

   Note that $-j\omega t - \alpha t$ does not represent a phase angle because it contains a real part ($-\alpha t$).

   The magnitude in this case ($Be^{-\alpha t}$) is time-varying, which is okay.

Find real and imaginary parts:

1. $\text{Re}\{8e^{-j0.12}\} = \text{Re}\{7.94 - j0.096\} = 7.94$

   The result $7.94 - j0.096$ is from the earlier polar-to-rectangular problem set.

   $\text{Im}\{8e^{-j0.12}\} = \text{Im}\{7.94 - j0.096\} = -0.096$

2. $\text{Re}\{14\angle 132°\} = \text{Re}\{-9.37 + j10.4\} = -9.37$

   The result $-9.37 + j10.4$ is from the earlier polar-to-rectangular problem set.

   $\text{Im}\{14\angle 132°\} = \text{Im}\{-9.37 + j10.4\} = 10.4$

3. $\text{Re}\{2\} = \text{Re}\{2 + j0\} = 2$
   $\text{Im}\{2\} = 0$

4. $\text{Re}\{j15\} = \text{Re}\{0 + j15\} = 0$
   $\text{Im}\{j15\} = 15$

5. $\text{Re}\{\sin\omega t\} = \text{Re}\{\sin\omega t + j0\} = \sin\omega t$
   $\text{Im}\{\sin\omega t\} = 0$

6. $\text{Re}\{j\cos\omega t\} = \text{Re}\{0 + j\cos\omega t\} = 0$
   $\text{Im}\{j\cos\omega t\} = \cos\omega t$

7. $\text{Re}\{x^2 + y^2 + j2xy\} = x^2 + y^2$
   $\text{Im}\{x^2 + y^2 + j2xy\} = 2xy$

Express in proper rectangular form:

1. $j5(-3 + j20) = (j5)(-3) + (j5)(j20) = -j15 - 100 = -100 - j15$

   Proper form is to put the real part on the left and the imaginary part on the right.

2. $-16(j2)^2 - j8(j3)^2 = -16(j^2)(2^2) - j8(j^2)(3^2) = -16(-1)(4) - j8(-1)(9) = 64 + j72$

3. $(j\beta + \alpha)^2 = (j\beta + \alpha)(j\beta + \alpha) = (j\beta)(j\beta) + 2(j\beta\alpha) + \alpha^2$
$= -\beta^2 + j2\alpha\beta + \alpha^2 = \alpha^2 - \beta^2 + j2\alpha\beta$

The real part is $\alpha^2 - \beta^2$, and the imaginary part is $2\alpha\beta$.

Find magnitudes and phases:

1. $e^{-j\pi/2} = 1e^{-j\pi/2}$

By inspection, the magnitude is 1 ($|e^{-j\pi/2}| = 1$),
and the phase is $-\pi/2$ (or $3\pi/2$, since $e^{-j\pi/2} = e^{j3\pi/2}$).

2. $5e^{-j\pi}$

By inspection, the magnitude is 5 ($|5e^{-j\pi}| = 5$),
and the phase is $-\pi$ (or $\pi$).
Also, note that $5e^{-j\pi} = 5e^{j\pi} = 5(-1) = -5$.

3. $-9e^{-j\pi/3} = 9(-1)e^{-j\pi/3} = 9e^{j\pi}e^{-j\pi/3} = 9e^{j(\pi-\pi/3)} = 9e^{j2\pi/3}$
or $-9e^{-j\pi/3} = 9e^{-j\pi}e^{-j\pi/3} = 9e^{j(-\pi-\pi/3)} = 9e^{-j4\pi/3}$
$\rightarrow \left|-9e^{-j\pi/3}\right| = 9$ and
phase $\left\{-9e^{-j\pi/3}\right\} = 2\pi/3$ or $-4\pi/3$

Note that the angle $2\pi/3$ ($120°$) is equivalent to the angle $-4\pi/3$ ($-240°$).

4. $|2 + j3| = \sqrt{(2)^2 + (3)^2} = 3.61$
phase$\{2 + j3\} = \tan^{-1}(3/2) = 0.983$ ($56.3°$)

Thus, $2 + j3 = 3.61e^{j0.983} = 3.61\angle 56.3°$

5. $|-j8| = \sqrt{(0)^2 + (-8)^2} = 8$
phase$\{-j8\} = \tan^{-1}(-8/0) = \tan^{-1}(-\infty) = -\pi/2$ ($-90°$)

6. $|jFe^{j0.83\pi}| = |j||F|\,|e^{j0.83\pi}|$

(The magnitude of a product is equal to the product of the magnitudes.)

Since $|j| = 1$ and $|e^{j0.83\pi}| = 1$, then
$|jFe^{j0.83\pi}| = |F| = F$

To find the phase, put the expression into proper polar form (using $j = e^{j\pi/2}$):

$jFe^{j0.83\pi} = Fe^{j\pi/2}e^{j0.83\pi} = Fe^{j(\pi/2+0.83\pi)} = Fe^{j1.33\pi}$
$\rightarrow$ phase$\{jFe^{j0.83\pi}\} = 1.33\pi$ $(239.4°)$ by inspection

The phase can also be expressed as $1.33\pi - 2\pi = -0.67\pi$ $(-120.6°)$.

7. The solution is very similar to that of Problem 6.
$|jFe^{jg}| = |j||F|\,|e^{jg}| = (1)(F)(1) = F$

To find the phase, put the expression into proper polar form:

$jFe^{jg} = Fe^{j\pi/2}e^{jg} = Fe^{j(\pi/2+g)}$
$\rightarrow$ phase$\{jFe^{jg}\} = \pi/2 + g$ by inspection

8. $|-15| = 15$

Put the expression into proper polar form:

$-15 = 15(-1) = 15e^{\pm j\pi}$
$\rightarrow$ phase$\{-15\} = \pi$ or $-\pi$ $(\pm 180°)$

9. $j = e^{j\pi/2}$, so $|j| = 1$

The phase is $\pi/2$ by inspection.

Find complex conjugates:

To find the complex conjugate of a complex number or expression, simply replace every occurrence of $j$ with $-j$. The conjugate of a sum is equal to the sum of the conjugates, and the conjugate of a product is equal to the product of the conjugates.

1. $j^* = -j$

2. $(2 + j7 - Fe^{jg})^* = 2 - j7 - Fe^{-jg}$

3. $(3 \times 10^{-6})^* = 3 \times 10^{-6}$

Real numbers are unaffected by complex conjugation.

4. $[Re\,\{5 + j15\}]^* = (5)^* = 5$

5. $[Im\,\{5 + j15\}]^* = (15)^* = 15$

6. $\left[-j832e^{-0.3z}e^{(7-j18)t}\right]^* = j832e^{-0.3z}e^{(7+j18)t}$

7. $[8e^{-j0.12}(0.3 + j6.1)]^* = 8e^{j0.12}(0.3 - j6.1)$