Hint for Graded Prob. 3 of HW #1

Some students are obtaining a value for R' that is half of that given in the posted answer for Graded Prob. 3. It occurs when the relationship

$$Z_0\gamma = \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}}\sqrt{(R'+j\omega L')(G'+j\omega C')} = R'+j\omega L'$$

is applied to solve the problem, which seems like a reasonable approach. However, the strategy does not work in this case because the value given for Z_0 in the problem statement is only approximate. Because the line is slightly lossy, Z_0 has a complex value with a very small imaginary part. It turns out that the missing imaginary part must be included if the expression above is to be used to find an accurate value for R'. The notes below explain why.

Start by obtaining alternate expressions for γ and Z_0 under the condition (applicable in Prob. 3) that $G' = 0$:

$$\gamma = \sqrt{(R'+j\omega L')(j\omega C')} = \sqrt{j\omega R'C' - \omega^2 L'C'} = \sqrt{-\omega^2 L'C' + j\omega R'C'}$$

and

$$Z_0 = \sqrt{\frac{R'+j\omega L'}{j\omega C'}} = \sqrt{\frac{R'}{j\omega C'}+\frac{L'}{C'}} = \sqrt{\frac{L'}{C'} - j\frac{R'}{\omega C'}}\,.$$

Factor out the real part of each expression:

$$\gamma = \sqrt{-\omega^2 L'C'}\sqrt{1 - j\frac{\omega R'C'}{\omega^2 L'C'}} = j\omega\sqrt{L'C'}\sqrt{1-j\frac{R'}{\omega L'}}$$

and

$$Z_0 = \sqrt{\frac{L'}{C'}}\sqrt{1-j\frac{R'}{\omega C'}\cdot\frac{C'}{L'}} = \sqrt{\frac{L'}{C'}}\sqrt{1-j\frac{R'}{\omega L'}}\,.$$

Multiplying the two expressions gives the same result as that shown at the top of this page:

$$\begin{aligned} Z_0\gamma &= \sqrt{\frac{L'}{C'}}\sqrt{1-j\frac{R'}{\omega L'}}\left(j\omega\sqrt{L'C'}\right)\sqrt{1-j\frac{R'}{\omega L'}} = j\omega\sqrt{L'C'}\sqrt{\frac{L'}{C'}}\left(1-j\frac{R'}{\omega L'}\right) \\ &= j\omega L'\left(1-j\frac{R'}{\omega L'}\right) = j\omega L' + R'. \end{aligned}$$

(*continued on next page*)

But now consider what happens when we approximate Z_0 as purely real (as is done in Prob. 3), that is, when we assume that

$$Z_0 = \sqrt{\frac{L'}{C'}}\,.$$

Forming the product $Z_0\gamma$ now yields

$$Z_0\gamma = \sqrt{\frac{L'}{C'}}\left(j\omega\sqrt{L'C'}\right)\sqrt{1-j\frac{R'}{\omega L'}} = j\omega\sqrt{L'C'}\sqrt{\frac{L'}{C'}}\sqrt{1-j\frac{R'}{\omega L'}}$$
$$= j\omega L'\sqrt{1-j\frac{R'}{\omega L'}}.$$

As in Graded Prob. 5, we can apply the approximation $(1 \pm x)^{1/2} \approx 1 \pm x/2$, which is valid for $|x| << 1$, because in this case $R' << \omega L'$ (the line is slightly lossy). The expression for $Z_0\gamma$ becomes

$$Z_0\gamma \approx j\omega L'\left(1-j\frac{R'}{2\omega L'}\right) = j\omega L' + \frac{R'}{2}.$$

The only difference between this expression for $Z_0\gamma$ and the previous one is that Z_0 is assumed to be purely real in this one. That assumption leads to a value for the real part of $Z_0\gamma$ that is half the value of R'. Thus, the product $Z_0\gamma$ cannot be used to find R' if Z_0 is assumed to be real or, alternatively, if the approximation is used, then the relationship $\text{Re}\{Z_0\gamma\} = 0.5R'$ must be applied.

To solve Graded Prob. 3 of HW #1, I would like you to apply a different approach to find R'. You may use the result(s) from one of the other problems if you wish.