ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Homework Assignment #1 – due via Moodle at 11:59 pm on Thursday, Sept. 21, 2023
[Prob. 5d revised 9/20/23]

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the solutions when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problems.

***Graded Problems*:**

1. A *permutation matrix* is a square matrix consisting of only zeros and ones that can rearrange the rows of another matrix by pre-multiplying it or the columns by post-multiplying it.

   a. Verify that the matrix P_{12} below interchanges rows 1 and 2 of the 3 × 3 matrix A.
   b. Find a permutation matrix P_{13} that interchanges rows 1 and 3 of the 3 × 3 matrix A.

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

2. Use Example 10 in Sec. 8.2 of Zill, 6th ed. as a guide to balance the chemical equation below.

$$Ca_3(PO_4)_2 + H_3PO_4 \rightarrow Ca(H_2PO_4)_2$$

3. [adapted from Problem 16 from Zill, 6th ed., Sec. 8.3] Let A be a nonzero 4 × 6 matrix.

   a. Find the maximum rank that A can have.
   b. Suppose that rank(A|$\mathbf{b}$) = 2. Find the values(s) of rank(A) for which the system $A\boldsymbol{x} = \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, is inconsistent.
   c. Find the values(s) of rank(A) for which the system is consistent.
   d. Suppose that rank(A) = 3. Find the number of parameters that the solution of the system $A\boldsymbol{x} = \mathbf{0}$ must have.

(*continued on next page*)

4. [adapted from Problem 17 from Zill, 6th ed., Sec. 8.3] Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the first, second, and third column vectors, respectively, of the matrix

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 1 & 0 & 2 \\ -1 & 5 & 13 \end{bmatrix}$$

Note that $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. What does this result imply about the returned value of the operation rank(A)? *Hint*: Read the "Remarks" at the end of Sec. 8.3.

5. An elementary row operation (ERO) that adds a multiple α of the j^{th} row of an $N \times N$ matrix A to its k^{th} row can be expressed as a matrix multiplication MA, where the matrix M is given by

$$M\left(j, k, \alpha\right) = I + \alpha \mathbf{e}_k \mathbf{e}_j^T,$$

where $\mathbf{e}_j$ is an $N \times 1$ column vector with a 1 in the j^{th} entry and zeros everywhere else. Note that the $\mathbf{e}$ vectors are orthogonal (i.e., $\mathbf{e}_j^T\mathbf{e}_k = 0$) and that $k \neq j$ in the expression above (i.e., there is no reason to add a multiple of a row to itself).

   a. Write out the full matrix for $M(2, 4, 3)$.
   b. Show that the inverse of $M(j, k, \alpha)$ is given by $M^{-1}\left(j, k, \alpha\right) = I - \alpha \mathbf{e}_k \mathbf{e}_j^T$.
   c. Verify the expression for the inverse in part b by finding the inverse of $M(2, 4, 3)$ directly either by hand or by using your calculator or software.
   d. **[text in boldface revised 9/20/23]** This type of ERO does not change the determinant of the matrix that it is applied to. That is, $|M(j, k, \alpha)\, A| = |A|$, so the determinant of $M(j, k, \alpha)$ must be 1. **Show that $|M(j, k, \alpha)| = 1$.**

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions (if applicable). Try not to give up too quickly if you struggle to solve any of them. Move on to a different problem and then come back to the difficult one after some time has passed.

Try working through all or most of Problems 34 through 44 at the end of Sec. 8.6 of Zill, 6th ed. They are all "show that…" problems, so you should be able to verify on your own whether you have solved them correctly.