ENGR 695 Advanced Topics in Engineering Mathematics Spring 2021

Homework Assignment #2 – due via Moodle at 11:59 pm on Friday, Sept. 29, 2023

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the solutions when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problems.

***Graded Problems*:**

1. The coefficients c_1 through c_3 of the linear model given below right that provide the best fit (in the least-squares sense) to the given set of *x*-*y* data can be found using the normal equation $F\mathbf{c} = \mathbf{y}$. Find the matrix F (i.e., show what its entries must be). Also use *Matlab* or other software to find the condition number of the matrix F^TF. Based on your experience with the previous lab exercises, would you expect the calculated coefficient values to be smooth, oscillatory, or perhaps somewhere in between?

x_i	y_i
0.001	1.1
0.002	5.3
0.003	8.8
0.004	13.6
0.005	22.1
0.006	15.3
0.007	10.8
0.008	5.2

$$y(x) = c_1 + c_2 x^2 + c_3 x e^x$$

2. Compute the eigensystem (i.e., the eigenvalues and eigenvectors) for the following matrices:

$$\text{a.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{b.} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{c.} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Make sure that you can do the calculations manually, but feel free to use the *Matlab* function `eig` to check your work.

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3. Show that the eigenvalues for the following matrix are $e^{\pm i\theta}$.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Then, using Euler's identity ($e^{i\theta} = \cos\theta + i\sin\theta$), show that the eigenvector associated with $\lambda_1 = e^{i\theta}$ can be expressed as shown below. As with the previous problem, you may use the *Matlab* function `eig` to check your work.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

4. Show or otherwise demonstrate that the following statements about eigenvalues are true (assume that all matrices are square):

a. The eigenvalues of A^{-1} are the reciprocals of the eigenvalues of the non-singular matrix A.
b. The eigenvalues of A and A^T are the same.
c. A matrix is singular if it has a zero eigenvalue.
d. The eigenvalues of a triangular matrix are the diagonal elements of that matrix.
e. For a non-singular matrix A, the eigenvalues of A^n, μ_k, are the eigenvalues of A raised to the power of n (i.e., $\mu_k = \lambda_k^n$).

5. Answer true or false to the following questions about eigensystems. Support your answers with reasons, examples, counterexamples, definitions, or other "proofs" as needed.

a. A matrix cannot have 0 as an eigenvalue.
b. A matrix is singular if it has one (or more) eigenvalues that are 0.
c. A symmetric matrix cannot be singular.
d. A matrix is singular if it has a repeated eigenvalue.

6. Show that if A is an $N \times N$ non-singular, symmetric matrix, then it may be written in the form

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \cdots + \lambda_N \mathbf{x}_N \mathbf{x}_N^T \,,$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, etc. are the eigenvectors of the matrix. *Hint*: Pay special attention to the special features of the eigensystems of symmetric matrices.

This form of representing A in terms of its eigensystem is sometimes referred to as the *spectral form* because it decomposes A into separate parts, each of which is based on a different component or characteristic of the eigensystem.

7. Find the rank of a matrix formed via the outer product $\mathbf{v}\mathbf{v}^T$, where $\mathbf{v}$ is a vector. Based on your answer, does the expression for A found in the previous problem seem odd or peculiar? Why or why not? *Hint*: If A is an $N \times N$ non-singular, symmetric matrix, then it is full rank.

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8. For each of the matrices below, determine how many of the singular values would be zero if the matrix were decomposed using the SVD method. Explain how you arrived at your answer. You may use *Matlab* or other software to check your answers, but your solution to each part must indicate that you understand how to find the number of singular values.

$$\text{a. } \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \qquad \text{b. } \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix} \qquad \text{c. } \begin{bmatrix} 2 & 1 & 1 \\ -4 & -2 & -2 \\ 8 & 4 & 4 \end{bmatrix} \qquad \text{d. } \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Determine which of the following expressions can be used as, or transformed into, an expression for a linear model meant to a fit a set of *x*-*y* data. Briefly show how you would generate the *F* (sampled function) matrix if it is possible. If not, explain why. In each case, the coefficients to be determined are indicated as c_1 and/or c_2. *Example*: If $y(x) = c_1x^n$, then the data transformations $Y = \ln(y)$ and $X = \ln(x)$ lead to the linear model $Y = nX + \ln(c_1)$ so that an *X*-*Y* plot of the data provides *n* as the slope and $\ln(c_1)$ as the vertical intercept.

a. $y = c_1 e^{c_2 x}$

b. $y = c_1 e^{-x} + c_2 x e^{-2x}$

c. $y = \dfrac{c_1 x}{1 + c_2 x}$

d. $y = c_1 \sin\left(c_2 x\right)$

2. In Lab #3, we saw that a type of constrained optimization works by minimizing the magnitudes of the coefficients calculated by the normal equation. The applicable cost function and modified normal equations are

$$E = \left|F\mathbf{c} - \mathbf{y}\right|^2 + \gamma\mathbf{c}^T\mathbf{c} \qquad \text{and} \qquad \mathbf{c} = \left(F^TF + \gamma I_N\right)^{-1} F^T\mathbf{y}\,.$$

We also saw that the second finite differences of the coefficients can be minimized using the modified normal equation

$$\left(F^TF + \gamma H\right)^{-1}\mathbf{c} = F^T\mathbf{y}\,,$$

where *H* is an $N \times N$ symmetric matrix related to the differencing operation. The latter set of equations is obtained by minimizing the cost function

$$E = \left|F\mathbf{c} - \mathbf{y}\right|^2 + \gamma\mathbf{c}^T H\mathbf{c}\,.$$

Using the derivation shown in the supplemental reading “Constrained Least-Squares Optimization Using Minimized Coefficient Magnitudes” as a guide, show how the second cost function above (with *H*) reduces to the second set of normal equations. You do not have to repeat the part of the derivation involving the $\left|F\mathbf{c} - \mathbf{y}\right|^2$ term.