ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Homework Assignment #4 – due via Moodle at 11:59 pm on Friday, Oct. 27, 2023
[Prob. 3 revised 10/26/23]

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

Unless otherwise specified, you may use *Matlab* or other mathematic software (or a calculator) to check your work.

It is your responsibility to review the solutions to the graded and ungraded problems when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problems.

***Graded Problems*:**

1. Find the eigenvalues and eigenfunctions of the boundary value problem given below. Consider the conditions $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

$$y'' + \lambda y = 0 \text{ with } y(0) = 0 \text{ and } y'(\pi/2) = 0$$

2. Find solutions to the following boundary value problems (BVPs).

a. $x^2 y'' + xy' + a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 1$
b. $x^2 y'' + xy' - a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 1$
c. $x^2 y'' + xy' + a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 0$
d. $x^2 y'' + xy' - a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 0$

3. **[text in boldface changed 10/26/23]** Find the eigenvalues and eigenfunctions of the boundary value problem given below. Also specify the form of the inner product used to test the orthogonality of the solutions $\{y_n(x)\}$.

$$x^2 y'' + xy' + \lambda y = 0 \text{ with } \lambda = \alpha^2,\ \alpha > 0 \text{ and BCs } \boldsymbol{y(1) = 0} \text{ and } \boldsymbol{y(5) = 0}$$

4. Show that the boundary value problem in the previous problem has only the trivial solution if $\lambda < 0$.

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5. Use the separation of variables (SOV) method to find, if possible, a set of ordinary differential equations (ODEs) that could be used to develop product solutions to the following partial differential equations (PDEs). For the problem that involves more than two independent variables, you will need more than one separation constant. You do not have to find the solutions to the ODEs.

a. $\dfrac{\partial^2 E}{\partial x^2}+\dfrac{\partial^2 E}{\partial y^2}+\dfrac{\partial^2 E}{\partial z^2}=0$ (Use separation constants k_x^2, k_y^2, and k_z^2)

b. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial x\,\partial y}+\dfrac{\partial^2 u}{\partial y^2}=0$

c. $y\dfrac{\partial^2 u}{\partial x\,\partial y}+u=0$

d. $k\dfrac{\partial^2 u}{\partial x^2}-u=\dfrac{\partial u}{\partial t}$, where k is a constant and $k > 0$

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. The norm and square norm of a function are defined by the expressions given below left. The square norm is also the self-product of a function. Find the square norm of the Fourier eigenfunction $y_n(x) = \cos(n\pi x)$, $n = 0, 1, 2, \ldots$ over the interval [0, 1] using analytical means. Provide a general result that applies for all values of n. You may look up the solution in an integral table to check your answer, but you must show the solution. *Hint*: Use a trigonometric identity or the identity given below right.

norm: $\|y_n\|=\sqrt{\int_a^b y_n^2(x)\,dx}$

square norm: $\|y_n\|^2=\int_a^b y_n^2(x)\,dx$

$\cos(x)=\dfrac{e^{ix}+e^{-ix}}{2}$

2. Chebyshev's equation is given below. For appropriate boundary conditions, it leads to an eigenvalue problem with orthogonal solutions $\{T_n(x)\}$ for $n = 0, 1, 2, \ldots$ that correspond to the eigenvalues $\{\lambda_n\}$. The solutions take the form of polynomials of order n; that is, as n increases, so does the order of the polynomial. Find the interval of x over which the solutions are orthogonal, and specify the form of the inner product used to test the orthogonality of the solutions $\{T_n(x)\}$.

$$\left(1-x^2\right)y''-xy'+n^2y=0$$

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3. The textbook (Zill, 6th ed.) presents a proof for the orthogonality of the nontrivial solutions to a Sturm-Liouville problem by starting with the self-adjoint form for two different eigenvalues and eigenfunctions given by

$$\frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right]+q(x)y_m+\lambda_m p(x)y_m=0$$

$$\frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right]+q(x)y_n+\lambda_n p(x)y_n=0$$

and then stating that multiplying the first equation by y_n and the second by y_m, subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields the expression

$$\begin{aligned}(\lambda_m-\lambda_n)\int_a^b p(x)y_m(x)y_n(x)\,dx = r(b)\left[y_m(b)y_n'(b)-y_n(b)y_m'(b)\right]\\ -r(a)\left[y_m(a)y_n'(a)-y_n(a)y_m'(a)\right].\end{aligned}$$

Work through the missing steps to prove the orthogonality condition above.

4. Use the separation of variables (SOV) method to find, if possible, a solution to the following PDE.

$$\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial u}{\partial x}=0$$