ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Homework Assignment #6 – due via Moodle at 11:59 pm on Friday, Nov. 10, 2023

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

Unless otherwise specified, you may use *Matlab* or other mathematic software (or a calculator) to check your work.

It is your responsibility to review the solutions to the graded and ungraded problems when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problems.

***Graded Problems*:**

1. [adapted from Prob. 18 of Sec. 14.2 of Zill, 6th ed.] The problem below describes the displacement u of a circular membrane with radius c that is vibrating in two dimensions. Complete the four parts listed below to find the SoV solution $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$.

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial^2 u}{\partial t^2}, \qquad 0 \le r \le c \quad \text{and} \quad t \ge 0$$

$$u(c,\theta,t)=0, \quad 0 \le \theta \le 2\pi \quad \text{and} \quad t \ge 0$$

$$u(r,\theta,0)=f(r,\theta), \quad 0 \le r \le c \quad \text{and} \quad 0 \le \theta \le 2\pi$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r,\theta), \qquad 0 \le r \le c \quad \text{and} \quad 0 \le \theta \le 2\pi .$$

a. Using the separation constants $-\lambda$ and $-\nu$, show that the separated ODEs are

$$T''+a^2\lambda T=0, \qquad \Theta''+\nu\Theta=0, \qquad \text{and} \qquad r^2R''+rR'+\left(\lambda r^2-\nu\right)R=0.$$

b. With $\lambda = \alpha^2$ and $\nu = \beta^2$, solve the separated equations in part a, and determine the eigenvalues and eigenfunctions of the problem.

c. Use the superposition principle to find a solution that involves a double sum. You do not have to find integral expressions for the coefficients.

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2. [adapted from Prob. 9 of Sec. 14.2 of Zill, 6th ed.] The temperature in a circular plate of radius *c* can be determined via the boundary value problem described below. The origin of the coordinate system, where *r* = 0, is at the center of the plate. Solve for *u*(*r*, *t*), including finding the expressions needed to determine the coefficients in the series expansion.

$$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}, \qquad 0\le r\le c \quad \text{and} \quad t\ge 0$$

$$u(c,t)=0, \quad t\ge 0 \quad \text{and} \quad u(r,0)=f(r), \quad 0\le r\le c$$

3. Find the forward, backward, and centered difference approximations to the first derivative of the functions shown below at the indicated points using the indicated intervals. Then find the centered difference approximation to the second derivative. Finally, find the percentage error of each derivative relative to the actual derivative obtained via analytical evaluation.

a. $f(t)=2.4\cos(120\pi t)$ at *t* = 4.0 ms with Δ*t* = 1.0 ms

b. $f(x)=\sqrt{x-2}$ (principal value) at *x* = 2.5 m with Δ*x* = 10 cm

***Ungraded Problems*:**

1. Find integral expressions for the four coefficients in the double sum in Graded Problem 1, which considers a circular membrane that is vibrating in two dimensions.

2. [adapted from Prob. 11 of Sec. 14.2 of Zill, 6th ed.] When there is heat transfer from the lateral side of an infinite circular cylinder of radius 1 (see figure below) into a surrounding medium at a temperature of 0 K, the temperature inside the cylinder is determined via the boundary value problem described below. The origin of the coordinate system, where *r* = 0, is at the center of the plate. Solve for *u*(*r*, *t*), including finding the expression needed to determine the coefficients in the series expansion.

$$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}, \qquad 0\le r\le 1 \quad \text{and} \quad t\ge 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1}=-hu(1,t), \qquad h>0 \quad \text{and} \quad t\ge 0$$

$$u(r,0)=f(r), \quad 0\le r\le c$$

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3. D'Alembert's solution to the one-dimensional wave equation in unbounded media, defined by

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad -\infty \le x \le \infty \quad \text{and} \quad t \ge 0$$

$$u(x,0) = f(x) \quad \left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0} = g(x)$$

is given by

$$u(x,t) = \frac{1}{2} f(x+at) + \frac{1}{2} f(x-at) + \frac{1}{2a} \int_{x-at}^{x+at} g(s)\,ds\,.$$

Consider a vibrating string problem in which the dependent variable u represents the displacement of the string from its equilibrium position, and $a = 300$ m/s, $f(x) = 0$, and $g(x)$ is a rectangular pulse function given by

$$g(x) = \mathrm{Rect}(x) = \begin{cases} A, & -0.25\text{ m} \le x \le 0.25\text{ m} \\ 0, & \text{elsewhere} \end{cases}$$

where $A = 900$ m/s. This simulates what would happen if someone were to strike the string with a 50 cm wide flat object. Plot the solution $u(x, t)$ over the spatial interval $x = -2$ m to $x = 2$ m at the times $t = 0.5$, 1, 2, and 4 ms. You will probably want to use mathematical analysis software like *Matlab* or *Mathematica* to produce the plots.

Hint: Consider evaluating the integral in the solution for six different cases ($r = 0.25$ m, the half-width of the object that strikes the string):

$x - at < -r$ and $x + at < -r$	$-r < (x - at) < r$ and $-r < (x + at) < r$
$x - at < -r$ and $-r < (x + at) < r$	$-r < (x - at) < r$ and $x + at > r$
$x - at < -r$ and $x + at > r$	$x - at > r$ and $x + at > r$