ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Homework Assignment #8 – due via Moodle at 11:59 pm on Friday, Dec. 1
[Probs. 1 revised at 5:30 pm on 12/1/23]

You may make reasonable assumptions and approximations to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work. You may use *Matlab* or other software to check your work.

It is your responsibility to review the solutions to the graded and ungraded problems when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

There are no ungraded problems in this homework set.

***Graded Problems*:**

1. **[minus signs added to 2nd equation below at 5:30 pm 12/2/23]** A Robin boundary condition is one that incorporates both the dependent variable and its first spatial derivative; it can be thought of as a linear combination of the Dirichlet and Neumann boundary conditions. A common application in heat equation problems arises when an object is in contact with a medium (such as the ground) that is held at a constant temperature. The transfer of heat is proportional to the difference in temperature between the end of the object at the boundary and the temperature u_m of the surrounding medium. If the boundary is located at $x = a$, the Robin boundary condition can be expressed as

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = h\left[u(a,t) - u_m\right] \quad \rightarrow \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} - h\,u(a,t) = -h\,u_m, \quad h > 0,$$

   where h is a constant and where the expression on the right places the derivative and the undifferentiated dependent variable to the left of the equal sign and the constant to the right. Note that the boundary condition is nonhomogeneous if $u_m \neq 0$. Find the modified update equation that must be applied at the boundary at $x = a$ to incorporate the Robin boundary condition into the explicit finite difference solution of the 1-D heat equation. Center the finite difference at location index $i = 1$, which corresponds to $x = a$. You do not have to find the modified update equation that is applicable at the other boundary (at $x = b$).

2. Repeat the previous problem for the Crank-Nicholson method described in Sec. 16.2 of the textbook (D. G. Zill, *Advanced Engineering Mathematics*, 6th ed.). That is, find the modified first equation in the system of equations used to compute u at each time step in the algorithm so that the Robin boundary condition described in the previous problem is properly incorporated. The solution space is discretized as $x = a + (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, where N_x is the total number of points. The time index is j, where $j = 0$ corresponds to $t = 0$. You do not have to find the modified equation that is applicable at the other boundary (at $x = b$).

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3. Neumann boundary conditions can also be incorporated into the solution of the one-dimensional heat equation problem using the Crank-Nicholson method. Assume that the spatial domain extends from $x = a$ to $x = b$ and that the boundary conditions are given by

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=b} = u_{xb},$$

where u_{xa} and u_{xb} are constants. The solution space is discretized as $x = a + (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, where N_x is the total number of points. The time index is j, where $j = 0$ corresponds to $t = 0$. Show that the first and last equations in the system of equations needed to compute u at each time step with the boundary conditions shown above are given by

$$-\alpha u_{1,j+1} + 2u_{2,\,j+1} = \beta u_{1,\,j} - 2u_{2,\,j} + 4\Delta x\, u_{xa} \quad \text{and} \quad 2u_{N_x-1,\,j+1} - \alpha u_{N_x,j+1} = -2u_{N_x-1,\,j} + \beta u_{N_x,j} - 4\Delta x\, u_{xb},$$

where

$$\alpha = 2\left(1 + \frac{\Delta x^2}{c\Delta t}\right) \quad \text{and} \quad \beta = 2\left(1 - \frac{\Delta x^2}{c\Delta t}\right).$$