ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Nov. 10

1. Unfinished business: Animation of vibrating drumhead (wave equation in cylindrical coordinates)
2. Finite difference example: For $f(x) = e^x$, approximate $f'(1.2)$ using centered finite differences with Δx = 0.1, 0.05, and 0.01. Exact result (to 5 sig. digs.) is $f'(1.2) = 3.3201$.

   a. $\Delta x = 0.1$: $f'(1.2) \approx \dfrac{f(1.2+0.05) - f(1.2-0.05)}{0.1} = \dfrac{e^{1.25} - e^{1.15}}{0.1} = 3.3215$

   b. $\Delta x = 0.05$: $f'(1.2) \approx \dfrac{f(1.2+0.025) - f(1.2-0.025)}{0.05} = \dfrac{e^{1.225} - e^{1.175}}{0.05} = 3.3205$

   c. $\Delta x = 0.01$: $f'(1.2) \approx \dfrac{f(1.2+0.005) - f(1.2-0.005)}{0.01} = \dfrac{e^{1.205} - e^{1.195}}{0.01} = 3.3201$

3. Application: Finite difference solution of the heat equation

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{where } c = \text{thermal diffusivity}$$

   a. Questions:
      i. What do finite difference approximations look like when there is more than one independent variable?
      ii. How many solution points (in x and in t) do we select?
      iii. What do we do about the boundaries?

   b. Finite difference approximations of partial derivatives (hold nondifferentiated variable constant)

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \quad \text{and} \quad \frac{\partial(x,t)}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

   c. Note that the x-derivative is approximated using a centered difference and the time derivative by a forward difference. We will see why very soon.

   d. Heat equation expressed using finite differences

$$c\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

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e. Need to define discrete points in space and time at which the dependent variable *u* is calculated. The arrays of points are called spatial and time *grids* or *meshes*.
   i. Solution space is along *x*-axis between boundaries $x = a$ and $x = b$.
   ii. Time assumed to begin at $t = 0$.
   iii. Space and time are discretized into N_x and N_t points, respectively:

$$x_i = a + (i-1)\Delta x, \quad i = 1, 2, 3, \ldots, N_x \quad \text{where} \quad \Delta x = \frac{b-a}{N_x - 1}$$

$$t_j = j\Delta t, \quad j = 0, 1, 2, 3, \ldots, (N_t - 1)$$

f. There is a constraint on Δt (examined soon).
g. Finite difference subscript notation:

$$u(x,t) = u_{i,j} \qquad u(x+\Delta x,t) = u_{i+1,j} \qquad u(x-\Delta x,t) = u_{i-1,j} \qquad u(x,t+\Delta t) = u_{i,j+1}$$

$$c\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \quad \rightarrow \quad c\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

h. Four of the five terms in FD form of equation are defined at time *t* (index *j*), but one is defined at time $t + \Delta t$ (index $j + 1$). Isolate that term on the left-hand side and move the rest to the right-hand side to form an *update equation*:

$$u_{i,j+1} - u_{i,j} = \frac{c\Delta t}{\Delta x^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] \quad \rightarrow \quad u_{i,j+1} = \frac{c\Delta t}{\Delta x^2}u_{i+1,j} + \left(1 - 2\frac{c\Delta t}{\Delta x^2}\right)u_{i,j} + \frac{c\Delta t}{\Delta x^2}u_{i-1,j}$$

i. This is an *explicit* FD method. The newest value of *u* at location *i* depends only on previous values and no values at other locations at the new time. That is, there is only one term at time index $j + 1$. A system of simultaneous equations is not required.
   i. This works because the time derivative was approximated using a forward difference.
   ii. Backward difference in time leads to *u* values evaluated at multiple adjacent locations at the same time, which would require a matrix solution. (Try it!)
   iii. Centered difference is troublesome because it requires a "look-back" in time. Starting the solution at $t = 0$ is challenging (how to handle $t - \Delta t$ term?).

$$c\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} = \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t}$$

4. Boundary and initial conditions

   a. Dirichlet BCs are simple: $u(a,t) = u_{1,j} = u_a$ and $u(b,t) = u_{N_x j} = u_b$, where u_a and u_b are constants (zero for homogeneous BCs)
   b. Neumann BCs are more challenging (later)
   c. Initial condition: $u(x,0) = u_{i,0} = f(x_i)$