ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Wednesday, Oct. 11

1. Inner products of functions and orthogonality – an important ODE/PDE solution tool

   a. Development: inner (dot) product of 3-D vectors and *N*-dimensional vectors…

   $$\mathbf{f}\cdot\mathbf{g}=\mathbf{f}^T\mathbf{g}=f_x g_x+f_y g_y+f_z g_z \qquad \text{and} \qquad \mathbf{f}^T\mathbf{g}=\sum_{j=1}^{N} f_j g_j$$

   inner product of functions (can be thought of as ∞-dimensional vectors)…

   $$\mathbf{f}^T\mathbf{g} \quad \rightarrow \quad \lim_{N\to\infty}\frac{b-a}{N}\sum_{j=1}^{N} f\left(x_j\right)g\left(x_j\right)=\int_a^b f\left(x\right)g\left(x\right)dx$$

   b. One common notation: $\left\langle f,g\right\rangle$ denotes an inner product. Context does the rest.

$\left\langle f,g\right\rangle=\mathbf{f}\cdot\mathbf{g}$	3-D vectors
$\left\langle f,g\right\rangle=\mathbf{f}^T\mathbf{g}$	*N*-dimensional vectors
$\left\langle f,g\right\rangle=\int_a^b f\left(x\right)g\left(x\right)dx$	functions

   c. The next step: If *f* and *g* are functions and $\left\langle f,g\right\rangle=0$, then the functions *f* and *g* are orthogonal. Yes, functions can be orthogonal too.

   d. Orthogonality makes possible the *practical* solution of many ODEs and PDEs

2. Example: Inner products of trigonometric (circular) functions

   a. Eigenfunctions of the Fourier equation (and the wave equation PDE, as we will see soon) over a bounded interval [−*L*, *L*] could be

   $$y_{1n}\left(x\right)=\cos\left(\frac{n\pi x}{L}\right) \qquad \text{and/or} \qquad y_{2n}\left(x\right)=\sin\left(\frac{n\pi x}{L}\right)$$

   b. Consider inner product (allow eigenvalues to be different; i.e., use *m* and *n*):

   $$\left\langle y_{1m},y_{2n}\right\rangle=\int_{-L}^{L} y_{1m}\left(x\right)y_{2n}\left(x\right)dx=\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

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c. Useful trigonometric identities:

$$\cos(a)\sin(b) = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$$
$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$
$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

d. Inner product becomes:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(m+n)\pi x}{L}\right]dx - \frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(m-n)\pi x}{L}\right]dx$$

e. Since $\sin(\alpha x)$ is an odd function, if $m \neq n$, then both integrals evaluate to zero. If $m = n$, then

$$\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{2n\pi x}{L}\right]dx - \frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(0)\pi x}{L}\right]dx = 0 + 0 = 0$$

f. Since $\langle y_{1m}, y_{2n}\rangle = 0$, the functions $y_{1m}(x)$ and $y_{2n}(x)$ are orthogonal for $m \neq n$ and for $m = n$.

g. What about the same eigenfunction with different eigenvalues ($m \neq n$)?

$$\begin{aligned}\langle y_{1m}, y_{1n}\rangle &= \int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(m+n)\pi x}{L}\right]dx + \frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(m-n)\pi x}{L}\right]dx \\ &= 0 + 0 \quad \text{(orthogonal)}\end{aligned}$$

h. …and the same eigenfunction with the same eigenvalues (self-product)?

$$\begin{aligned}\langle y_{1m}, y_{1n}\rangle &= \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx \\ &= \int_{-L}^{L}\left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right]dx = \frac{1}{2}\int_{-L}^{L}dx + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2n\pi x}{L}\right)dx \\ &= L + 0 = L \quad \text{(not orthogonal; self-product is square of "length")}\end{aligned}$$

i. Norm and square norm (general definitions):

$$\|y_n\| = \sqrt{\int_a^b y_n^2(x)\,dx} \quad \text{and} \quad \|y_n\|^2 = \int_a^b y_n^2(x)\,dx$$