ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Wednesday, Nov. 15

1. Finite difference solution of the wave equation

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

a. Easy to replace partial derivatives with centered differences

$$v^2 \left[\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \right] = \frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{\Delta t^2}$$

b. Form spatial and time grids (solution points). Use same approach as with heat equation:

$$x_i = a + (i-1)\Delta x, \quad i = 1, 2, 3, \ldots, N_x \quad \text{where} \quad \Delta x = \frac{b-a}{N_x - 1}$$

and

$$t_j = j\Delta t, \quad j = 1, 2, 3, \ldots, N_t$$

c. FD approximation of the wave equation becomes

$$v^2 \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \right] = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta t^2}$$

d. Solve for $u_{i,j+1}$ to obtain explicit update equation:

$$u_{i,j+1} = \left(\frac{v\Delta t}{\Delta x} \right)^2 u_{i+1,j} + 2\left[1 - \left(\frac{v\Delta t}{\Delta x} \right)^2 \right] u_{i,j} + \left(\frac{v\Delta t}{\Delta x} \right)^2 u_{i-1,j} - u_{i,j-1}$$

or

$$u_{i,j+1} = C_1 u_{i+1,j} + C_2 u_{i,j} + C_3 u_{i-1,j} - u_{i,j-1},$$

where $C_1 = C_3 = \left(\frac{v\Delta t}{\Delta x} \right)^2$ and $C_2 = 2\left[1 - \left(\frac{v\Delta t}{\Delta x} \right)^2 \right]$

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e. How do we handle the $u_{i,j-1}$ term at $j = 0$ ($t = 0$) given that u is not defined for $t < 0$? ($j = -1$ corresponds to time that precedes $t = 0$) We derive a special update equation!

Use the time derivative initial condition for the special case of $j = 0$ (first time step)

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad \rightarrow \quad \left.\frac{\partial u}{\partial t}\right|_{t=0,x=x_i} = g(x_i) \quad \rightarrow \quad \frac{u_{i,1} - u_{i,-1}}{2\Delta t} \approx g(x_i) \quad \rightarrow \quad u_{i,-1} = u_{i,1} - 2\Delta t\, g(x_i).$$

Regular update equation evaluated at $j = 0$: $\quad u_{i,1} = C_1 u_{i+1,0} + C_2 u_{i,0} + C_3 u_{i-1,0} - u_{i,-1}$

Substitute FD approximation of IC into update equation evaluated at $j = 0$:

$$u_{i,1} = C_1 u_{i+1,0} + C_2 u_{i,0} + C_3 u_{i-1,0} - \left[u_{i,1} - 2\Delta t\, g(x_i)\right]$$

Simplify to obtain special update equation for first time step:

$$u_{i,1} = \frac{C_1}{2} u_{i+1,0} + \frac{C_2}{2} u_{i,0} + \frac{C_3}{2} u_{i-1,0} + \Delta t\, g(x_i)$$

f. Stability condition (sometimes called the Courant-Levy condition)

$$\frac{v\Delta t}{\Delta x} \le 1 \quad \rightarrow \quad \Delta t \le \frac{\Delta x}{v}$$

But also note that

$$\frac{v\Delta t}{\Delta x} \le 1 \quad \rightarrow \quad v \le \frac{\Delta x}{\Delta t} \quad \rightarrow \quad v \le v_g,$$

where v_g = "grid speed." The actual wave should not be able to "outrun" the solution. The "grid speed" can be thought of as a speed limit on the actual waves that are simulated. Put another way, the solution needs to be able to "keep up" with the actual waves being simulated.

2. FD solutions of wave equation: computational considerations

   a. Accuracy generally improves as spatial step size Δx and/or time step size Δt is decreased, although not always (e.g., CFL condition is most accurate setting for Δt in explicit FD solution of wave equation)
   b. Grid dispersion: artificial change in velocity (usually frequency dependent) due to discretization in space; can reduce by using small spatial step sizes
   c. Grid dissipation: artificial attenuation (usually frequency dependent) due to discretization in space; can reduce by using small spatial step sizes
   d. Finite precision of computer representation of numbers becomes a problem for very small step sizes

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3. FD solutions in non-Cartesian coordinate systems

    a. Challenging due to more complicated expressions and variable step sizes
    b. Often better to use Cartesian system and then apply a staircase approximation to irregular boundaries or interfaces between materials

4. Multiple materials in solution space:

    a. Additional interior boundary conditions might be necessary at interfaces
    b. Some FD methods inherently account for interior boundaries