ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Oct. 16

1. Important general form of ODE: the regular Sturm-Liouville equation

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right]+q(x)y+\lambda p(x)y=0$$

subject to homogeneous boundary conditions defined over interval $[a, b]$:

$$A_1 y(a)+B_1 y'(a)=0, \quad A_1 \text{ and } B_1 \text{ not both zero}$$
$$A_2 y(b)+B_2 y'(b)=0, \quad A_2 \text{ and } B_2 \text{ not both zero}$$

   a. some textbooks use $p(x)$ for $r(x)$ and $w(x)$ for $p(x)$
   b. second-order with variable coefficients
   c. form above is called "self-adjoint" form
   d. λ is a parameter in the problem (eigenvalue)
   e. $r(x)$, $p(x) > 0$ on interval of solution
   f. importance for our purposes:
      i. guarantees orthogonality, completeness, representation (see supplemental reading "Sturm-Liouville Problems: Eigenfunction Orthogonality")
      ii. function $p(x)$ defines kernel for inner product

2. The "Sturm-Liouville Insurance Policy" (SLIP)

   a. There are non-trivial solutions for specific values of the parameter λ (eigenvalues).
   b. There is an infinity of eigenvalues.
   c. There is a smallest but not a largest eigenvalue.
   d. The eigenvalues are real and distinct ($\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$).
   e. For each eigenvalue there is a single solution (eigenfunction) $y_n(x)$
   f. The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$. That is,

$$\langle y_m, y_n\rangle_{p(x)}=\int_a^b y_m(x)y_n(x)p(x)\,dx=\begin{cases}0, & m\neq n\\ C_m, & m=n\end{cases}$$

   g. The set of solutions to an S-L problem are complete in that the set forms a basis for the space of square-integrable functions on the interval $[a, b]$.

$$f(x)=\sum_{n=1}^{\infty}a_n y_n(x) \quad \text{with} \quad a_n=\frac{\langle f(x), y_n(x)\rangle}{\langle y_n(x), y_n(x)\rangle}$$

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Proof:

Multiply both sides of weighted sum expression by $y_m(x)$ and evaluate the inner product. Don't forget the "weight" function $p(x)$:

$$\int_a^b f(x)\, y_m(x)\, p(x)\, dx = \sum_{n=1}^{\infty} a_n \int_a^b y_m(x)\, y_n(x)\, p(x)\, dx$$

Because of the SLIP, we know that the eigenfunctions $y_n(x)$ are orthogonal. Thus,

$$\int_a^b y_m(x)\, y_m(x)\, p(x)\, dx = 0 \text{ for } m \neq n,$$

so

$$\int_a^b f(x)\, y_m(x)\, p(x)\, dx = a_m \int_a^b y_m(x)\, y_m(x)\, p(x)\, dx$$

Expressed in inner product notation,

$$\langle f(x), y_m(x) \rangle = a_m \langle y_m(x), y_m(x) \rangle \quad \rightarrow \quad a_m = \frac{\langle f(x), y_m(x) \rangle}{\langle y_m(x), y_m(x) \rangle}$$

The expression for a_n on the previous page follows after changing the index from m to n.

3. The SLIP is such a valuable set of properties that it is worth determining whether a given problem is a Sturm-Liouville problem. Examples:

   a. Is $y'' + \lambda y = 0$ a S-L equation?
   b. Is $y'' - \lambda y = 0$ a S-L equation?
   c. Is $y'' - \lambda x y = 0$ a S-L equation?
   d. Is $x^2 y'' + x y' + \left(\lambda x^2 - \nu^2\right) y = 0$ a S-L equation?
   e. Is $a(x) y'' + b(x) y' + c(x) y + \lambda d(x) y = 0$ a S-L equation?

4. Converting second-order ODEs to self-adjoint Sturm-Liouville form

   a. A second-order ODE of the form

$$a(x) y'' + b(x) y' + c(x) y + \lambda d(x) y = 0$$

   can be converted to the equivalent Sturm-Liouville equation in adjoint form

$$\frac{d}{dx}\left[r(x) \frac{dy}{dx} \right] + q(x) y + \lambda p(x) y = 0 .$$

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b. Conversion steps:

   i. Compute the integrating factor $\mu(x)$ (watch out for $a(x) = 0$ for any x over the bounded interval):

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right)$$

   ii. Compute the elements of the S-L adjoint form:

$$r(x) = \mu(x)$$

$$q(x) = \frac{c(x)}{a(x)} \mu(x)$$

$$p(x) = \frac{d(x)}{a(x)} \mu(x)$$

   iii. Verify that $r(x), p(x) > 0$ on interval of solution

5. Example: Convert parametric Bessel's equation to Sturm-Liouville equation in self-adjoint form:

$$x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$$

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp\left(\int \frac{x}{x^2} dx\right) = \exp\left(\int \frac{1}{x} dx\right) = e^{\ln x} = x$$

$$\rightarrow \quad r(x) = \mu(x) = x \qquad q(x) = \frac{-\nu^2}{x^2} x = \frac{-\nu^2}{x} \qquad p(x) = \frac{x^2}{x^2} x = x\,.$$

Self-adjoint form of Bessel's equation:

$$\frac{d}{dx}\left[x \frac{dy}{dx}\right] - \frac{\nu^2}{x} y + \lambda x y = 0$$

Significance: We now know the kernel $p(x)$ used in the inner product; $p(x) = x$.