ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Wednesday, Oct. 18

1. Converting second-order ODEs to self-adjoint Sturm-Liouville form

   a. A second-order ODE of the form

   $$a(x)\,y'' + b(x)\,y' + c(x)\,y + \lambda d(x)\,y = 0$$

   can be converted to the equivalent Sturm-Liouville equation in adjoint form

   $$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + q(x)\,y + \lambda\,p(x)\,y = 0\,.$$

   b. Conversion steps:

      i. Compute the integrating factor $\mu(x)$ (watch out for $a(x) = 0$ for any x over the bounded interval):

      $$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)}\,dx\right)$$

      ii. Compute the elements of the S-L adjoint form:

      $$r(x) = \mu(x)$$

      $$q(x) = \frac{c(x)}{a(x)}\,\mu(x)$$

      $$p(x) = \frac{d(x)}{a(x)}\,\mu(x)$$

      iii. Verify that $r(x)$, $p(x) > 0$ over interval of solution

2. Example: Convert parametric Bessel's equation to Sturm-Liouville equation in self-adjoint form:

   $$x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$$

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Identify the variable coefficients:

$$a(x) = x^2 \qquad b(x) = x \qquad c(x) = -\nu^2 \qquad d(x) = x^2$$

Compute the integrating factor:

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp\left(\int \frac{x}{x^2} dx\right) = \exp\left(\int \frac{1}{x} dx\right) = e^{\ln x} = x$$

$$\rightarrow \quad r(x) = \mu(x) = x \qquad q(x) = \frac{c(x)}{a(x)}\mu(x) = \frac{-\nu^2}{x^2}x = \frac{-\nu^2}{x} \qquad p(x) = \frac{d(x)}{a(x)}\mu(x) = \frac{x^2}{x^2}x = x$$

Self-adjoint form of Bessel's equation:

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] - \frac{\nu^2}{x}y + \lambda xy = 0$$

Significance: We now know the kernel $p(x)$ used in the inner product; $p(x) = x$:

$$\langle y_m, y_n \rangle = \int_a^b x y_m(x) y_n(x) dx = \begin{cases} 0, & m \neq n \\ C_m, & m = n \end{cases}$$

3. Example: Solve the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

a. Compare to general form $x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0 \rightarrow \nu = 0$
b. Solution:

$$y(x) = c_1 J_0\left(\sqrt{\lambda}x\right) + c_2 Y_0\left(\sqrt{\lambda}x\right) \quad \rightarrow \quad y'(x) = -c_1\sqrt{\lambda}J_1\left(\sqrt{\lambda}x\right) - c_2\sqrt{\lambda}Y_1\left(\sqrt{\lambda}x\right)$$

Apply BC #1:

$$y'(0) = 0 = -c_1\sqrt{\lambda}J_1(0) - c_2\sqrt{\lambda}Y_1(0) = -c_1\sqrt{\lambda}(0) - c_2\sqrt{\lambda}(-\infty)$$

Since $Y_1(0) \rightarrow -\infty$ (as does $Y_0(0)$), $Y_0\left(\sqrt{\lambda}x\right)$ is not a viable solution. Apply BC #2:

$$y(1) = 0 = c_1 J_0\left(\sqrt{\lambda}\right)$$

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This implies that λ can only have values for which $\sqrt{\lambda_n} = r_n$, n = 1, 2, 3, …, where r_n are the roots (zeros) of J_0.

c. First four roots of J_0: 2.4048, 5.5201, 8.6537, 11.7915

d. Try evaluating inner product with and without $p(x)$; this is the focus of Lab #6:

$$\left\langle J_0\left(\sqrt{\lambda_m}x\right), J_0\left(\sqrt{\lambda_n}x\right)\right\rangle = \int_0^1 J_0\left(\sqrt{\lambda_m}x\right) J_0\left(\sqrt{\lambda_n}x\right) dx = ?$$

or

$$\left\langle J_0\left(\sqrt{\lambda_m}x\right), J_0\left(\sqrt{\lambda_n}x\right)\right\rangle = \int_0^1 x J_0\left(\sqrt{\lambda_m}x\right) J_0\left(\sqrt{\lambda_n}x\right) dx = ?$$