ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Sept. 1

1. Solvability of M-by-N systems (see flowchart in Fig. 8.3.1)

   a. M = no. of equations; N = no. of unknowns
   b. $r = \text{rank}(A) = \text{rank}(A|\mathbf{b})$: consistent & solution possible
      i. $r = N$: unique solution
      ii. $r < N$: infinitely many solutions
   c. $r = \text{rank}(A) < \text{rank}(A|\mathbf{b})$: inconsistent & no solution possible
   d. One more thing: $\text{rank}(A) = \text{rank}(A^T)$

2. Triangulation example:

   a. boat at location (x_o, y_o) – two unknowns x_o and y_o
   b. direction finders at locations (x_1, y_1) and (x_2, y_2) – coordinates are known
   c. θ = angle to boat relative to axis joining the direction finders
   d. System of equations:

$$\begin{bmatrix} \sin\theta_1 & -\cos\theta_1 \\ \sin\theta_2 & -\cos\theta_2 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} x_1 \sin\theta_1 - y_1 \cos\theta_1 \\ x_2 \sin\theta_2 - y_2 \cos\theta_2 \end{bmatrix}$$

   e. If $\theta_1 \neq \theta_2$, then rank = 2 and boat can be located. If $\theta_1 = \theta_2$, then rank = 1 and location is indeterminate. Latter case occurs if boat lies along line joining the two direction finders.

3. Generality #1: Overdetermined systems are usually (but not always) inconsistent.

   Examples: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

   Although an overdetermined system might not have an exact solution, it could still have a “best” approximate solution.

4. Generality #2: Underdetermined systems are usually (but not always) consistent:

   Examples: $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

   Underdetermined systems can have infinitely many solutions or no solution but never a unique solution because $\text{rank}(A) \leq M < N$ always.

(*continued on next page*)

5. New topic: Curve-fitting and the method of least squares

   a. Start with an example. Consider the following small data set. How can we estimate the value of $y(3)$, that is, the value of y at $x = 3$?

i	x_i	y_i
1	1.0	1.1
2	2.0	3.2
3	4.0	5.2

   b. One possible approach: Set up a matrix expression that computes the coefficients of the quadratic expression for a curve that passes through the data points.

$$y = c_0 + c_1 x + c_2 x^2$$

   Is the matrix equation solvable?
   If so, is the solution acceptable?

   c. Another possible approach: Set up a matrix expression that computes the coefficients of the linear expression for a line that passes through the data points.

$$y = c_0 + c_1 x$$

   Is the matrix equation solvable?
   If so, is the solution acceptable?

   d. Next: a general approach applicable to any set of functions or curves