ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Oct. 20

1. Orthogonality conditions on solutions to Sturm-Liouville problem

   a. Consider the Sturm-Liouville equation in self-adjoint form for two different eigenvalues

$$\frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right]+q(x)y_m+\lambda_m p(x)y_m=0$$

$$\frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right]+q(x)y_n+\lambda_n p(x)y_n=0$$

   b. Multiplying the first equation by y_n and the second by y_m, subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields

$$(\lambda_m-\lambda_n)\int_a^b p(x)y_m(x)y_n(x)dx=r(b)\left[y_m(b)y_n'(b)-y_n(b)y_m'(b)\right]$$
$$-r(a)\left[y_m(a)y_n'(a)-y_n(a)y_m'(a)\right]$$

   c. Note that the left-hand side includes the inner product. One implication of this result is that the boundary conditions must be homogenous if the solutions y_m and y_n are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero. (See item #4 below.)

   d. Another implication is that y_m and y_n can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous.

2. Singular Sturm-Liouville problem

   a. Addresses cases when $r(x) > 0$ is not satisfied at one or both boundaries

   b. Right-hand side of equation in item 1b above is zero when:

      i. $r(a) = 0$ and $y_m(b)y_n'(b)-y_n(b)y_m'(b)=0$
      ii. $r(b) = 0$ and $y_m(a)y_n'(a)-y_n(a)y_m'(a)=0$
      iii. $r(a) = r(b) = 0$ and no BCs are specified at $x = a$ or $x = b$
      iv. $r(a) = r(b)$ and the BCs are $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic BCs)
      v. Caveat: The solutions $\{y_n\}$ are orthogonal if $r(a) = 0$ and/or $r(b) = 0$ provided that the solutions are bounded at the corresponding boundary.

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3. Example: Recall the parametric Bessel's equation

$$x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$$

Conversion to Sturm-Liouville equation in self-adjoint form yields $r(x) = x$, so $r(0) = 0$. We considered the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

General solution is

$$y(x) = c_1 J_0\left(\sqrt{\lambda} x\right) + c_2 Y_0\left(\sqrt{\lambda} x\right),$$

but this is a singular S-L problem because $r(0) = 0$. Also, because $Y_0(0) \to -\infty$, $Y_0\left(\sqrt{\lambda} x\right)$ is not a viable solution. However, we can show that

$$y_m(1) y_n'(1) - y_n(1) y_m'(1) = (0) y_n'(1) - (0) y_m'(1) = 0$$

because the second BC $y(1) = 0$ applies to all solutions. Thus, there are nontrivial, orthogonal solutions to this BVP.

4. Note that

$$A_1 y_m(a) + B_1 y_m'(a) = 0 \quad \rightarrow \quad A_1 y_m(a) = -B_1 y_m'(a)$$

$$A_1 y_n(a) + B_1 y_n'(a) = 0 \quad \rightarrow \quad A_1 y_n(a) = -B_1 y_n'(a)$$

Dividing first equation by second (assuming that neither A_1 nor B_1 is zero) yields

$$\frac{y_m(a)}{y_n(a)} = \frac{y_m'(a)}{y_n'(a)} \quad \rightarrow \quad y_m(a) y_n'(a) - y_n(a) y_m'(a) = 0.$$

Also satisfied if either $A_1 = 0$ or $B_1 = 0$. For example, if $A_1 \neq 0$ and $B_1 = 0$, then $y_m(a) = 0$ and $y_n(a) = 0$, which still guarantees that $y_m(a) y_n'(a) - y_n(a) y_m'(a) = 0$.

Similar result for other general BC at $x = b$.