ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Oct. 23

1. New major topic: Partial differential equations (PDEs). Applications:

   a. Heat transfer
   b. Diffusion
   c. Wave propagation (acoustic, mechanical, electromagnetic, etc.)
   d. Acoustics
   e. Highway traffic
   f. Many, many others

2. General form and classification (significance of classification most apparent in Chap. 16):

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

   a. Hyperbolic: $B^2-4AC>0$
   b. Parabolic: $B^2-4AC=0$
   c. Elliptic: $B^2-4AC<0$

3. Types of PDEs on which we will concentrate and conditions

   a. 1-D heat equation: $k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \quad k\geq 0$ (parabolic)
   b. 2-D wave equation: $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$ (hyperbolic)
   c. 2-D Laplace's equation: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (elliptic)
   d. All three PDEs above can be extended to more physical dimensions
   e. All problems have boundary conditions and initial conditions
   f. Many are eigenvalue problems
   g. Later: Numerical solutions to PDEs

4. Method of separation of variables (SOV)

   a. Overall solution assumed to be product of solutions in each independent variable.
   b. Heat equation example:

Let $u(x,t)=X(x)T(t)$

(*continued on next page*)

Substitution leads to

$$k\frac{\partial^2(XT)}{\partial x^2}=\frac{\partial(XT)}{\partial t} \quad\rightarrow\quad kX''T=XT' \quad\rightarrow\quad k\frac{X''}{X}=\frac{T'}{T}$$

The goal is to make one side dependent on only one variable. In this case, the left-hand side depends only on *x*, and the right-hand side only on *t*.

c. Introduce a new quantity, the separation constant:

$$k\frac{X''}{X}=\frac{T'}{T}=-\lambda$$

Why does this work? Consider a third arbitrary independent variable; call it *z*. Take the derivative of the separated equation with respect to z:

$$\frac{\partial}{\partial z}\left(k\frac{X''}{X}\right)=\frac{\partial}{\partial z}\left(\frac{T'}{T}\right)=0$$

Because both derivatives are zero, each side of the separated equation must be equal to a constant.

d. Result is two linked ODEs:

$$X''+\lambda X=0 \quad \text{and} \quad T'+\lambda kT=0$$

e. Solution to ODE in one independent variable could be an infinite sum of eigenfunctions corresponding to different eigenvalues (superposition principle)

f. Two questions:
    i. Which ODE should be associated with constants in original PDE (e.g., the constant *k* in the heat equation example above)?
    ii. Why not use λ instead of $-\lambda$ for separation constant?

g. Not always possible to find a solution with this method; some PDE solutions are not separable