ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Sept. 25

1. Singular value decomposition (SVD): Implications and properties

   a. $A = U\Sigma V^T = \sum_{j=1}^{N} \sigma_j \mathbf{u}_j \mathbf{v}_j^T$, where $\mathbf{u}_j \mathbf{v}_j^T$ is an outer product, each of which is $M \times N$.
      i. The outer products have rank = 1. (See HW #2 Prob. 7.)
      ii. A is a weighted sum of rank-1 $M \times N$ matrices.
      iii. The weights (σ_j) grow progressively smaller.
      iv. If any of the singular values are zero or too small to matter (lost in the noise, for example), then A can be represented by two compact sets of orthogonal vectors $\{\mathbf{u}_i\}_{i = 1 \text{ to } r}$ and $\{\mathbf{v}_i\}_{i = 1 \text{ to } r}$, where $r < N$.
      v. The summation terms for zero or nearly zero singular values can be ignored. (Example given in image processing demo later.)

   b. Use complex conjugate transpose for complex matrix A.

   c. U and V are both orthogonal ($U^{-1} = U^T$ and $V^{-1} = V^T$); thus,
      $A = U\Sigma V^T \quad \rightarrow \quad AV = U\Sigma \quad \rightarrow \quad A\mathbf{v}_i = \sigma_i \mathbf{u}_i$

   d. $A = U\Sigma V^T \quad \rightarrow \quad A^T = V\Sigma^T U^T \quad \rightarrow \quad A^T U = V\Sigma \quad \rightarrow \quad A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i$

   e. $A^T A = \left(U\Sigma V^T\right)^T \left(U\Sigma V^T\right) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$
      $\rightarrow \quad \left(A^T A\right)V = V\Sigma^T\Sigma \quad \rightarrow \quad \left(A^T A\right)\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$
      σ_i^2 are the eigenvalues of $A^T A$, and $\{\mathbf{v}_i\}_{i = 1 \text{ to } N}$ are the eigenvectors

   f. $AA^T = \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$
      $\rightarrow \quad \left(AA^T\right)U = U\Sigma^T\Sigma \quad \rightarrow \quad \left(AA^T\right)\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i$
      σ_i^2 are also the eigenvalues of AA^T, and $\{\mathbf{u}_i\}_{i = 1 \text{ to } N}$ are the eigenvectors

   g. If A is symmetric, then $A^T A = AA^T = A^2$, so $\lambda_i^2 = \sigma_i^2 \quad \rightarrow \quad |\lambda_i| = |\sigma_i|$ (sign ambiguity)

2. Applications and examples

   a. To solve a system $A\mathbf{x} = \mathbf{b}$ (for overdetermined and square systems; underdetermined requires interpretation):

      $U\Sigma V^T \mathbf{x} = \mathbf{b} \quad \rightarrow \quad \Sigma V^T \mathbf{x} = U^T \mathbf{b} \quad \rightarrow \quad V^T \mathbf{x} = \Sigma^{-1} U^T \mathbf{b} \quad \rightarrow \quad \mathbf{x} = V\Sigma^{-1} U^T \mathbf{b}$

      Since Σ is diagonal (in the "economy-sized" decomposition),

(*continued on next page*)

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/\sigma_N \end{bmatrix}_{N\times N}$$

b. Example #1: Compare eigenvalues to singular values of symmetric matrix A (use *Matlab* `eig` and `svd` commands):

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

i. Notice order of singular values
ii. Check orthogonality of columns of U and V
iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
iv. What is the rank of this matrix?
v. What are the ranks of $\mathbf{u}_1\mathbf{v}_1^T$, $\mathbf{u}_2\mathbf{v}_2^T$, and $\mathbf{u}_3\mathbf{v}_3^T$?

c. Example #2: Compare eigenvalues to singular values of the singular matrix A:

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ 4 & 1 & -6 \end{bmatrix}$$

i. Notice order of singular values
ii. Check orthogonality of columns of U and V
iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
iv. What is the rank of this matrix?
v. What are the ranks of $\mathbf{u}_1\mathbf{v}_1^T$, $\mathbf{u}_2\mathbf{v}_2^T$, and $\mathbf{u}_3\mathbf{v}_3^T$?

d. Example #3: Image processing demonstration.