ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Nov. 27

1. Finite difference solution of the heat equation with Neumann BCs

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

   a. Dirichlet BCs are straightforward:

$$u(a,t) = u_{1,j} = u_a \quad \text{and} \quad u(b,t) = u_{N_x,j} = u_b, \quad \text{for } t \geq 0$$

   where u_a and u_b are usually constants but could be time varying; easily handled

   b. Neumann BCs (often used to model insulation at boundaries):

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=b} = u_{xb}, \quad \text{for } t \geq 0$$

   where u_{xa} and u_{xb} are usually constants but could be time varying

   c. One approach (for BC at $x = a$):

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} \approx \frac{u(a+\Delta x,t) - u(a-\Delta x,t)}{2\Delta x} = \frac{u_{2,j} - u_{0,j}}{2\Delta x} = u_{xa};$$

   double-sized interval ($2\Delta x$) does not add significant error; less error than forward or backward difference with interval Δx

   d. Note that $u_{0,j}$ (located at $x = a - \Delta x$) is outside solution space. Express it in terms of quantities that exist:

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa}$$

   e. Update equation that applies to interior points:

$$\text{general case: } u_{i,j+1} = C_1\, u_{i+1,j} + C_2\, u_{i,j} + C_3\, u_{i-1,j}$$

$$\text{for } i = 1\text{: } u_{1,j+1} = C_1\, u_{2,j} + C_2\, u_{1,j} + C_3\, u_{0,j}$$

$$\text{where } C_1 = C_3 = \frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad C_2 = 1 - 2\frac{c\Delta t}{\Delta x^2}$$

(*continued on next page*)

f. Substitute expression for $u_{0,j}$ into the general update equation evaluated at $i=1$:

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa} \quad \text{and} \quad u_{1,j+1} = C_1\, u_{2,j} + C_2\, u_{1,j} + C_3\, u_{0,j}$$

$$\rightarrow \quad u_{1,j+1} = C_1\, u_{2,j} + C_2\, u_{1,j} + C_3 \left(u_{2,j} - 2\Delta x\, u_{xa} \right)$$
$$\rightarrow \quad u_{1,j+1} = C_4\, u_{2,j} + C_2\, u_{1,j} - C_5\, u_{xa} \,,$$

$$\text{where} \quad C_4 = \frac{2c\Delta t}{\Delta x^2} \quad \text{and} \quad C_5 = 2\Delta x \frac{c\Delta t}{\Delta x^2} = \frac{2c\Delta t}{\Delta x}$$

a. Similar result for BC at $x = b$:

$$u_{N_x,j+1} = C_1 \left(u_{N_x-1,j} + 2\Delta x\, u_{xb} \right) + C_2\, u_{N_x,j} + C_3\, u_{N_x-1,j}$$

$$\rightarrow \quad u_{N_x,j+1} = C_4\, u_{N_x-1,j} + C_2\, u_{N_x,j} + C_5\, u_{xb}$$

b. These two special update equations are applied only at the boundaries.

2. Alternate approach for handling Neumann BCs

   a. Add "fictional" solution space points at $i = 0$ and $i = N_x + 1$
   b. Increase size of solution vector `u` by two (i.e., to $N_x + 2$); append solution values to beginning and end of vector. Could instead add two special variables to hold `u` at end points
   c. Update equations applied to end points (after interior points have been updated):

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa} \quad \text{and} \quad u_{N_x+1,j+1} = u_{N_x-1,j+1} + 2\Delta x\, u_{xb}$$

3. Crank-Nicholson Method (an implicit method) applied to heat equation

   a. Issues with explicit method just considered:
      i. centered difference for x-derivative and forward difference for t-derivative
      ii. mixed differencing reduces accuracy slightly for a given Δx
      iii. stability criterion limits size of Δt
   b. One alternative: center all derivatives at time $t + 0.5\Delta t$. FD approximations become

$$t\text{-derivative:} \quad \frac{\partial \left(x, t + 0.5\Delta t \right)}{\partial t} \approx \frac{u\left(x, t+\Delta t \right) - u\left(x, t \right)}{\Delta t}$$

$$x\text{-derivative:} \quad \frac{\partial^2 u\left(x, t+0.5\Delta t \right)}{\partial x^2} \approx \frac{u\left(x+\Delta x, t+0.5\Delta t \right) - 2u\left(x, t+0.5\Delta t \right) + u\left(x-\Delta x, t+0.5\Delta t \right)}{\Delta x^2}$$

(*continued on next page*)

c. Indexing doesn't allow half time-steps, so

$$\frac{\partial^2 u(x,t+0.5\Delta t)}{\partial x^2} \approx \frac{1}{2}\left[\frac{\partial^2 u(x,t+\Delta t)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial x^2}\right]$$

$$\approx \frac{1}{2}\left[\frac{u(x+\Delta x,t+\Delta t)-2u(x,t+\Delta t)+u(x-\Delta x,t+\Delta t)}{\Delta x^2}\right.$$

$$\left.+\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2}\right]$$

d. Approximation of x-derivative using index notation:

$$\frac{\partial^2 u(x,t+0.5\Delta t)}{\partial x^2} \approx \frac{1}{2\Delta x^2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

e. FD approximation of heat equation becomes

$$\frac{c}{2\Delta x^2}\left(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

f. Multiply both sides by $2\Delta x^2/c$, then gather $j + 1$ (new) terms on the left and j (old) terms on the right:

$$u_{i+1,j+1} - \left(2 + \frac{2\Delta x^2}{c\Delta t}\right)u_{i,j+1} + u_{i-1,j+1} = -u_{i+1,j} + \left(2 - \frac{2\Delta x^2}{c\Delta t}\right)u_{i,j} - u_{i-1,j}$$

Left-hand side has terms at three adjacent locations ($i + 1$, i, and $i - 1$), which leads to a set of coupled equations, that is, a system of equations (matrix equation).