ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Oct. 27

1. Finish heat equation example:

   a. PDE with BCs and IC

   $$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{with} \quad u(0,t) = 0, \quad u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x)$$

   b. Solution is

   $$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$$

   c. Determination of coefficients in summation:

   Apply IC: $f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

   Multiply by *m*th eigenfunction and weighting function [$p(x) = 1$ in this case] and integrate over interval of interest:

   $$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

   Because the $X(x)$ problem is an S-L problem, the eigenfunctions are orthogonal. Thus, the inner products for $m \neq n$ are zero, so

   $$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = A_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = A_m \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2m\pi x}{L}\right)\right]dx = A_m\left(\frac{L}{2}\right)$$

   $$\rightarrow \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

2. Interpretation of solution

   a. How do we break it down?
   b. Does it satisfy BCs?
   c. Does it make sense? Behavior as $t \to \infty$
   d. What can we learn from it?
   e. *Matlab* simulation

(*continued on next page*)

3. Not always possible to find a solution with this method; some PDE solutions are not separable. Which of the following PDEs can be solved via SOV, and which cannot?

   a. $x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$

   b. $a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}$, where g is a constant

   c. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + u$

   d. $a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}$