ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Sept. 29

1. Important theorems and concepts applicable to ODEs

   a. Linear N^{th} order differential equation (DE):

   $$a_n(x)\frac{d^n y}{dx^n}(x)+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_0(x)y=g(x)$$

   If $g(x) = 0$, then the DE is *homogenous*

   b. IVPs have unique solutions (not true for BVPs)
   c. Superposition principle; a linear combination of solutions to a homogeneous DE over an interval is also a solution over the same interval
   d. Corollaries: 1) A constant multiple of a solution is also a solution; 2) homogeneous DE always possess the trivial solution $y = 0$
   e. Linearly independent vs. linearly dependent solutions (analogy to vectors)
   f. An N^{th} order homogeneous linear DE has a fundamental set of N linear independent solutions. The general solution is

   $$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

   g. Nonhomogeneous DEs: general solution = complementary solution + particular solution (complementary solution is the full solution set of the corresponding homogenous DE)
   h. Superposition also applies to particular solutions: If y_{p1} is a solution of the DE with $g_1(x)$, y_{p2} is a solution with $g_2(x)$, etc., then $y_{p1} + y_{p2} + \ldots$ is a solution to

   $$a_n(x)\frac{d^n y}{dx^n}(x)+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_0(x)y=g_1(x)+g_2(x)+\cdots$$

2. Solution of nonhomogeneous linear ODEs with constant coefficients (not emphasized in course)

   a. You can guess or…
   b. Method of undetermined coefficients (Sec. 3.4 of Zill, 6th ed.) – doesn't work for all forcing functions
   c. Method of variation of parameters (Sec. 3.5 of Zill, 6th ed.) – more general & complicated
   d. Use the annihilator method (see web link) – annihilators do not always exist

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3. Boundary value problems (BVPs) involving special DEs

   a. Primarily concerned with 2nd order DEs (most common in mathematical physics)
   b. Fourier equation and modified Fourier equation
   c. Cauchy-Euler equation
   d. Bessel equation
   e. Others (Legendre, Airy, …)

4. Solutions to Fourier and modified Fourier equations:

$$y'' + a^2 y = 0 \qquad \text{and} \qquad y'' - a^2 y = 0$$

   a. For closed boundaries (i.e., problem defined over finite range of x), recommend

$$y(x) = c_1 \cos(ax) + c_2 \sin(ax) \qquad \text{and} \qquad y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

   Roots r_1 and r_2 of characteristic equation imaginary for Fourier equation and real for modified Fourier equation

   b. For open boundaries (i.e., problem defined over infinite or semi-infinite range of x), recommend

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

5. Example #1: BVP involving Fourier equation:

$$y'' + a^2 y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

   a. Nontrivial solution is

$$y(x) = c_2 \sin(n\pi x), \quad n = 1, 2, 3, \dots$$

   b. Infinitely many nontrivial solutions since infinitely many integers n will work. This is an eigenvalue problem. Constants $a_n = n\pi$ are *eigenvalues*, and elementary solutions $\sin(n\pi x)$ are *eigenfunctions*.
   c. The constant c_2 is left unspecified in this problem. However, if there had been a forcing function [i.e., $y'' + a^2 y = g(x)$], then c_2 could be uniquely specified.

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6. Example #2: Now consider an arbitrary value for a^2 but same BCs:

$$y'' + 2.5\pi y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

   a. No nontrivial solutions because $y(1) = 0$ is not satisfied; $y = 0$ is still a solution.

$$y(x) = c_2 \sin(n\pi x), \quad n = 1, 2, 3, \ldots$$

7. Example #3: BVP involving modified Fourier equation:

$$y'' - a^2 y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

   a. Attempt to apply

$$y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

   b. No nontrivial solutions because $y(1) = 0$ is not satisfied; $y = 0$ is still a solution.